Balancing domain decomposition by constraints algorithms for incompressible Stokes equations with nonconforming finite element discretizations

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Abstract

Hybridizable Discontinuous Galerkin (HDG) is an important family of methods, which combine the advantages of both Discontinuous Galerkin in terms of flexibility and standard finite elements in terms of accuracy and efficiency. The impact of this method is partly evidenced by the prolificacy of research work in this area. Weak Galerkin (WG) is a relatively newly proposed method by introducing weak functions and generalizing the differential operator for them. This method has also drawn remarkable interests from both numerical practitioners and analysts recently. HDG and WG are different but closely related. BDDC algorithms are developed for numerical solution of elliptic problems with both methods. We prove that the optimal condition number estimate for BDDC operators with standard finite element methods can be extended to the counterparts arising from the HDG and WG methods, which are nonconforming finite element methods. Numerical experiments are conducted to verify the theoretical analysis. Further, we propose BDDC algorithms for the saddle point system arising from the Stokes equations using both HDG and WG methods. By design of the preconditioner, the iterations are restricted to a benign subspace, which makes the BDDC operator effectively positive definite thus solvable by the conjugate gradient method. We prove that the algorithm is scalable in the number of subdomains with convergence rate only dependent on subdomain problem size. The condition number bound for the BDDC preconditioned Stokes system is the same as the optimal bound for the elliptic case. Numerical results confirm the theoretical analysis.

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Introduction

1.1 Overview

Numerical simulation of partial differential equations (PDEs) of practical interest usually lead to large scale linear systems with high condition number, and the solution of which can be a challenging task. Direct methods for such systems can be prohibitively expensive, if not impossible. Iterative methods, such as Krylov subspace methods, can be regarded as slow, as the number of iterations needed for convergence depends on the condition number. The construction of preconditioner is usually necessary to accelerate the convergence. Domain decomposition techniques provide powerful tools to construct preconditioners for large scale linear algebraic system obtained from the discretization of PDEs. In solving a boundary value problem, this type of methods divide the original problem into a number of smaller size problems over the subdomains, which are easier to solve. The convergence is achieved by iterating the subdomain local problems and/or a globally posed coarse problem.

Based on the degree of overlapping among subdomains, domain decomposition methods can be categorized as overlapping and non-overlapping domain decomposition methods. The former case is also referred to as Schwarz alternating method. In each Krylov iteration, the classical one-level method solve a local problem on each subdomain. The number of iterations are dependent on the number of subdomains. An advancement of this method is achieved by

introducing a second coarse level to the algorithm. The condition number of the two-level methods can be of the order $O(\frac{H}{\delta})$, where H and δ stand for the subdomain size and the size of the overlapping between subdomains, respectively. For a general introduction of the abstract theory of Schwarz methods, see (Toselli & Widlund, 2005, Chapter 2, 3).

The non-overlapping domain decomposition method is also called the iterative substructuring method. We can think of this class of methods as eliminating the interior variables of elements to some stage and solving the reduced linear system by a preconditioned Krylov subspace method(Toselli & Widlund, 2005). Two important families of the domain decomposition algorithms are the Neumann-Neumann and finite element tearing and interconnecting(FETI) methods. The Balancing Domain Decomposition by Constraints (BDDC) algorithm, introduced by Dohrmann for second order elliptic problem (Dohrmann, 2003), is a variant of the balancing Neumann-Neumann (BNN) algorithm, similarly as the dual-primal FETI (FETI-DP) to the FETI method. In the BDDC algorithm, the coarse problems are formed by a set of primal constraints on the interface. It has been proved for elliptic problem that the preconditioned operators for BDDC and FETI-DP have identical spectra except possibly at 0 and 1(Mandel et al., 2005; Li & Widlund, 2006b).

It is worth noting that the design of the efficient preconditioners are dependent both on the problem and the numerical discretization method. Different numerical methods will generate matrices of different properties such as sparsity patterns, which can be exploited in the design of efficient solution strategies of the linear system. The BDDC is one of the most advanced pre-conditioners in the field of domain decomposition. One advantage of the BDDC pre-conditioner over the more generic pre-conditioners such as incomplete LU (ILU) is that it does not require ad-hoc parameter tuning, as it is tailored to the discretized system by design. Extensive research has been conducted on BDDC for elliptic and Stokes problem solved using standard finite element methods, but relatively few has studied BDDC for non-conforming finite element. In this work, we will design BDDC pre-conditioners for the Poisson and Stokes problem discretized using the hybridizable discontinuous Galerkin

(HDG) and weak Galerkin (WG) methods, which are recently proposed non-conforming methods.

1.2 Functional Analysis Tools

1.2.1 Sobolev Spaces

We assume $\Omega \in \mathbb{R}^n$ is a bounded open set with Lipschitz continuous boundary. Thus the outward normal can be defined almost everywhere on $\partial\Omega$.

We start from the space of square integrable functions on Ω, i.e.,

$$L^2(\Omega) = \left\{ u : \Omega \mapsto \mathbb{R} \mid \int_\Omega |u|^2 dx < \infty \right\},$$

and define the subspace of $L^2(\Omega)$ with zero mean to be

$$L_0^2(\Omega) = \left\{ u \in L^2(\Omega) \mid \int_\Omega u dx = 0 \right\}.$$

Recall that Hilbert space is a vector space with topology defined by an inner product. $L^2(\Omega)$ is a Hilbert space with inner product and induced norm given by:

$$(u, v)_{L^2(\Omega)} = \int_\Omega uv dx; \quad \|u\|^2_{L^2(\Omega)} = \int_\Omega |u|^2 dx.$$

We introduce the multi-index notation for derivatives

$$D^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u \coloneqq \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$$

with $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The Sobolev space $H^k(\Omega)$ for any integer $k \geq 1$ is defined as the functional space such

that for each $u \in H^k(\Omega)$, there exists $u_\alpha \in L^2(\Omega)$ such that

$$\langle D^\alpha u, \phi \rangle = \int_\Omega u_\alpha \phi dx, \quad \phi \in C_0^\infty(\Omega)$$

for every multi-index α, with $\alpha \leq k$.

It is a Hilbert space with inner product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)},$$

an induced norm $\|\cdot\|_{H^k(\Omega)}$

$$\|u\|^2_{H^k(\Omega)} = (u, u)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^2 dx,$$

and a semi-norm

$$|u|^2_{H^k(\Omega)} = \sum_{|\alpha| = k} \int_\Omega |D^\alpha u|^2 dx.$$

We define $H_0^k(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$.

Let H_Ω be the diameter of Ω. We have the following scaled norm for the Sobolev space $H^1(\Omega)$:

$$\|u\|^2_{H^1(\Omega)} = \frac{1}{H_\Omega^2} \|u\|^2_{L^2(\Omega)} + |u|^2_{H^1(\Omega)} = \frac{1}{H_\Omega^2} \|u\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u|^2 dx,$$

with

$$\nabla = \mathbf{grad} = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right).$$

The space $H(div; \Omega)$ is the square integral vector-valued functions such that their divergence is also square integrable; i.e.,

$$H(div; \Omega) = \left\{ \mathbf{u} : \Omega \mapsto \mathbb{R}^n | \mathbf{u} \in [L^2(\Omega)]^n, \nabla \cdot \mathbf{u} \in L^2(\Omega) \right\},$$

with

$$div\mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}$$

where u_i is the i-th component of $\mathbf{u}$.

This is a Hilbert space with inner product and scaled graph norm defined by

$$(\mathbf{u}, \mathbf{v})_{H(div;\Omega)} = \int_\Omega \mathbf{u} \cdot \mathbf{v} dx + \int_\Omega div\mathbf{u}\, div\mathbf{v} dx, \quad \|u\|^2_{H(div;\Omega)} = \frac{1}{H^2_\Omega} \|u\|^2_{L^2(\Omega)} + \|\nabla \cdot \mathbf{u}\|^2_{L^2(\Omega)}.$$

1.2.2 Trace Spaces

In domain decomposition methods, we frequently need to work with some Sobolev spaces on a set $\Gamma \subseteq \partial\Omega$. The trace space $H^{1/2}(\partial\Omega)$ consists of functions on $\partial\Omega$ such that

$$\|u\|^2_{H^{1/2}(\partial\Omega)} = \frac{1}{H_{\partial\Omega}} \|u\|^2_{L^2(\partial\Omega)} + |u|^2_{H^{1/2}(\partial\Omega)} \leq \infty$$

with the seminorm

$$|u|^2_{H^{1/2}(\partial\Omega)} = \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^n} dxdy,$$

where $H_{\partial\Omega}$ is the diameter of $\partial\Omega$. These definitions can be generalized to a proper subset $\Gamma \subset \partial\Omega$ with positive $(n-1)$-dimensional measure and which is relatively open with respect to $\partial\Omega$.

The space $H_0^{1/2}(\partial\Omega)$ is defined as the closure of the space of $C_0^\infty(\partial\Omega)$. Note that $H_0^{1/2}(\partial\Omega)$, $H_0^{1/2}(\Gamma)$ coincides with $H^{1/2}(\partial\Omega)$, $H^{1/2}(\Gamma)$, respectively. We also define a proper subspace of $H^{1/2}(\Gamma)$ as

$$H_{00}^{1/2}(\Gamma) = \left\{ u \in H^{1/2}(\Gamma) | \mathcal{E}u \in H^{1/2}(\partial\Omega) \right\},$$

where $\mathcal{E}u$ is the extension by zero of u to $\partial\Omega$. The norm of $H_{00}^{1/2}(\Gamma)$ can be defined as

$$\|u\|^2_{H_{00}^{1/2}(\Gamma)} = \int_\Gamma \frac{u^2(x)}{d(x, \partial\Gamma)} dx + |u|^2_{H^{1/2}(\Gamma)},$$

where $d(x, \partial\Gamma)$ is the distance from x to the boundary of Γ. For $u \in H^{1/2}(\partial\Omega)$ which vanishes almost everywhere on $\partial\Omega \setminus \Gamma$, $\|u\|_{H^{1/2}(\partial\Omega)}$ and $\|u\|_{H_{00}^{1/2}(\Gamma)}$ are equivalent norms.

Note that the Green's formula can be generalized to functions in $[H^1(\Omega)]^n$.

Lemma 1.2.1. *For $\mathbf{u} \in [H^1(\Omega)]^n$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega (\nabla \cdot \mathbf{u})\, v dx + \int_\Omega \mathbf{u} \cdot \nabla v dx = \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n}) v dS.$$

1.2.3 Extension and Trace Theorems

We collect a few results on extension and trace theorems(Widlund, 2011; Tu, 2006).

For any Lipschitz domain Ω, there is a bounded extension operator

$$\mathcal{E}_\Omega : W_p^k(\Omega) \to W_p^k(\mathbb{R}^n),$$

such that

$$\|\mathcal{E}_\Omega u\|_{W_p^k(\mathbb{R}^n)} \leq C_\Omega \|u\|_{W_p^k(\Omega)}.$$

Here the spaces W_p^k are Sobolev spaces based on L^p. For $H^s(\Omega)$ function with $s < 1/2$, the extension by zero from Ω to $\mathbb{R}^n$ defines a bounded operator. Similarly, extension of $H^s(\Gamma)$, $\Gamma \subset \partial\Omega$ by zero to $\partial\Omega \setminus \Gamma$ also defines a bounded operator only for $s < 1/2$.

For a Lipschitz domain, it is easy to define the trace $\gamma_0 u$ of a smooth function u on the boundary $\partial\Omega$. γ_0 can be extended to all of $H^1(\Omega)$ and the range of this mapping is $H^{1/2}(\partial\Omega)$. The $|\cdot|_{H^{1/2}(\partial\Omega)}$ of an element g, e.g., Dirichlet data given on all of $\partial\Omega$, can be defined by $|\mathcal{H}g|_{H^1(\Omega)}$, where $\mathcal{H}$ is the harmonic extension into Ω.

Lemma 1.2.2 (Trace theorem)**.** *Let Ω be a Lipschitz domain. There exists a bounded linear operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ such that $\gamma_0 u = u\,|_{\partial\Omega}$ if u is continuous in $\bar{\Omega}$.*

Lemma 1.2.3 (Extension theorem)**.** *Let Ω be a Lipschitz domain. There exists a continuous lifting operator $\mathcal{L}_0 : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ such that $\gamma_0(\mathcal{L}_0 u) = u$, for $u \in H^{1/2}(\partial\Omega)$.*

1.2.4 Poincaré and Friedrichs Type Inequalities

Poincaré and Friedrichs type inequalities are important tools for the analysis of domain decomposition methods. We collect some results as related to this study. For details, refer to (Toselli & Widlund, 2005).

Lemma 1.2.4 (Poincaré Inequality)**.** *Let $u \in H^1(\Omega)$. Then, there exist constants, depending only on Ω, such that*

$$\|u\|^2_{L^2(\Omega)} \leq C_1 |u|^2_{H^1(\Omega)} + C_2 \left(\int_\Omega u dx \right)^2 .$$

Lemma 1.2.5 (Friedrichs Inequality)**.** *Let $\Gamma \subseteq \partial\Omega$ have nonvanishing $n-1$-dimensional measure. Then, there exist constants, depending only on Ω and Γ, such that, for $u \in H^1(\Omega)$,*

$$\|u\|^2_{L^2(\Omega)} \leq C_1 |u|^2_{H^1(\Omega)} + C_2 \|u\|^2_{L^2(\Gamma)} .$$

In particular, if u vanishes on Γ,

$$\|u\|^2_{L^2(\Omega)} \leq C_1 |u|^2_{H^1(\Omega)}$$

and thus

$$|u|^2_{H^1(\Omega)} \leq \|u\|^2_{H^1(\Omega)} \leq (C_1 + 1) |u|^2_{H^1(\Omega)} .$$

The following corollary can be obtained by simple scaling arguments.

Corollary 1.2.5.1. *Let Ω be Lipschitz continuous with diameter H. Then, there exists a constant $\hat{C}_1$, that depends only on the shape of Ω but not on its size, such that*

$$\|u\|^2_{L^2(\Omega)} \leq \hat{C}_1 H^2 |u|_{H^1(\Omega)} ,$$

for $u \in H^1(\Omega)$ with vanishing mean value on Ω. Similarly, if $\Gamma \subset \partial\Omega$ is defined as in Lemma

1.2.5 and has a diameter of order H, then

$$\|u\|^2_{L^2(\Omega)} \leq \hat{C}_2 H^2 \, |u|^2_{H^1(\Omega)} + \hat{C}_3 H \, \|u\|^2_{L^2(\Gamma)} \,,$$

for $u \in H^1(\Omega)$.

In the analysis of iterative substructuring methods, we need some inequalities involving functions on the boundary. The following results can be proved using the operators γ_0 and $\mathcal{L}_0$ of Lemmas 1.2.2 and 1.2.3.

Lemma 1.2.6. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz continuous polyhedron. If $u \in H^{1/2}(\partial\Omega)$ either has a vanishing mean value on $\partial\Omega$ or belongs to the closure of the space of $C^\infty(\partial\Omega)$ functions that vanish on a face of Ω, there exists a constant $\hat{C}_4$, that depends only on the shape of Ω but not on its size, such that*

$$\|u\|^2_{L^2(\partial\Omega)} \leq \hat{C}_4 H \, |u|^2_{H^{1/2}(\partial\Omega)} \,.$$

Similarly, if $F \subset \partial\Omega$ is one of the faces of Ω of diameter H and $u \in H^{1/2}(\mathcal{F})$ either has vanishing mean value on $\mathcal{F}$ or belongs to $H^{1/2}_{00}(\mathcal{F})$, then there exists a constant $\hat{C}_5$, that depends only on the shape of $\mathcal{F}$ but not on its size, such that

$$\|u\|^2_{L^2(\mathcal{F})} \leq \hat{C}_5 H \, |u|^2_{H^{1/2}(\mathcal{F})} \,.$$

1.3 Positive Definite Problems

Let V be a Hilbert space, $a(\cdot,\cdot)\colon V \times V \to \mathbb{R}$ a bounded, V-elliptic bilinear form and $l\colon V \to \mathbb{R}$ a bounded linear functional. The variational formulation for this elliptic problem is to find $u \in V$ such that

$$a(u,v) = l(v) = \langle l, v \rangle, \qquad v \in V. \tag{1.1}$$

Definition 1.3.1. *Bounded bilinear forms*

A bilinear form $a(\cdot,\cdot)\colon V\times V\to\mathbb{R}$ is said to be bounded, if there exists a constant $C\geq 0$ such that

$$|a(u,v)|\leq C\,\|u\|_V\,\|v\|_V\,,\qquad u,v\in V. \tag{1.2}$$

Definition 1.3.2. *V-elliptic bilinear forms*

A bilinear form $a(\cdot,\cdot)\colon V\times V\to\mathbb{R}$ is called V-elliptic (or, equivalently, coercive), if there exists a constant $\alpha>0$ such that

$$|a(u,u)|\geq \alpha\,\|u\|_V^2\,,\qquad u\in V. \tag{1.3}$$

The Lax-Milgram Lemma ensures the existence and uniqueness of the solution of (1.1).

Theorem 1.3.1. *Lax-Milgram Lemma Let V be a Hilbert space with dual V^* and assume that $a(\cdot,\cdot)\colon V\times V\to\mathbb{R}$ is a symmetric, bounded, and V-elliptic bilinear form and $l\in V^*$. Then, the variational equation (1.1) has a unique solution, satisfying*

$$\|u\|_V\leq\frac{1}{\alpha}\,\|l\|_{V'}\,,$$

where α is the coercivity constant.

1.4 Saddle Point Problems

Let V and Q be Hilbert spaces with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_Q$, and associated norms $\|\cdot\|_V$, $\|\cdot\|_Q$ and assume that $a(\cdot,\cdot)\colon V\times V\to\mathbb{R}$ and $b(\cdot,\cdot)\colon V\times Q\to\mathbb{R}$ are continuous bilinear forms. We denote by V^* and Q^* the dual spaces, and bounded linear functionals $f\in V^*$ and $g\in Q^*$. Consider the problem: find $(u,p)\in V\times Q$, such that

$$\begin{aligned} a(u,v)+b(v,p) &= \langle f,v\rangle_{V^*,V}, && v\in V,\\ b(u,q) &= \langle g,q\rangle_{Q^*,Q}, && q\in Q. \end{aligned} \tag{1.4}$$

We can associate continuous linear operators $A\colon V\to V^*$ and $B\colon V\to Q^*$ with the bilinear

forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, respectively, such that

$$\langle Au, v\rangle_{V^*,V} = a(u,v), \qquad u, v \in V,$$

$$\langle Bv, q\rangle_{Q^*,Q} = b(v,q), \qquad v \in V, q \in Q.$$

Thus the saddle point problem (1.4) can be rewritten as a system of operator equations as below

$$\begin{aligned} Au + B^*p &= f \quad in \quad V^*, \\ Bu &= g \quad in \quad Q^*. \end{aligned} \tag{1.5}$$

Theorem 1.4.1. *Existence and Uniqueness*

The saddle point problem (1.4) admits a solution $(u,p) \in V \times Q$, where $u \in V$ is uniquely determined and $p \in Q$ is unique up to an element of $Ker(B^*)$, if the following conditions hold for any $f \in V^*$ and $g \in Im(B)$:

- *The bilinear form $a(\cdot,\cdot)$ is $Ker(B)$-elliptic, i.e., there exists a constant $\alpha > 0$ such that*

$$a(v_0, v_0) \geq \alpha \|v_0\|_V^2, \qquad v_0 \in Ker(B);$$

- *The bilinear form $b(\cdot,\cdot)$ satisfies the Brezzi condition*

$$\inf_{q \in Q \setminus Ker(B^*)} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_{Q \setminus Ker(B^*)}} \geq \beta > 0.$$

The Brezzi condition is also known as inf-sup condition.

1.5 Finite Element Methods

The finite element method is a general technique to build finite-dimensional subspaces of a Hilbert space V in order to apply the Ritz-Galerkin method to a variational problem. The

test functions in the approximation subspace are usually chosen to be piecewise polynomials. Based on the inclusion property of the approximation subspaces, finite element methods subdivide into conforming or nonconforming methods (Brezzi & Fortin, 1991).

With respect to a triangulation $\mathcal{T}_h$ of the computational domain $\Omega \subset \mathbb{R}^d$, conforming finite elements are methods such that the resulting globally defined function obtained by summation of locally defined function over elements $K \in \mathcal{T}_h$ belongs to the underlying function space V for the variational formulation of the original partial differential equation(Brezzi & Fortin, 1991; Hoppe, 2016). In this context, the Ritz-Galerkin method seeks an approximate solution u_h in a suitable finite dimensional subspace V_h of V, i.e., $u_h \in V_h$, such that

$$a(u_h, v_h) = l(v_h), \qquad v_h \in V_h. \tag{1.6}$$

We recall some results from standard finite element methods (Brezzi & Fortin, 1991; Málek & Strakoš, 2015; Hoppe, 2016).

1.5.1 Approximation of Elliptic Problems

If $a(\cdot,\cdot)$ is a bounded, V-elliptic bilinear form, then by the Lax-Milgram Lemma (1.3.1), eqn. (1.6) admits a unique solution u_h. This solution is as good as the best approximation of $u \in V$ by a function in V_h, and the discretization error $u - u_h$ is bounded by the best approximation of solution $u \in V$ by functions in V_h, as given below.

Theorem 1.5.1. *Céa's Lemma*

Under the assumptions of the Lax-Milgram lemma, let $u \in V$ and $u_h \in V_h$ be the unique solutions of (1.1) and (1.6), respectively. Then, there holds

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V . \tag{1.7}$$

Céa's Lemma can be proved based on V-ellipticity and boundedness of $a(\cdot,\cdot)$, and the

following a-orthogonality property of the Galerkin method:

$$a(u-u_h, v_h) = 0, \qquad v_h \in V_h.$$

1.5.2 Approximation of Saddle Point Problems

We consider the approximation of (1.4) by finite dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$:

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= \langle f, v_h \rangle_{V^*,V}, \qquad v_h \in V_h, \\ b(u_h, q_h) &= \langle g, q_h \rangle_{Q^*,Q}, \qquad q_h \in Q_h. \end{aligned} \tag{1.8}$$

We denote by $A_h \colon V_h \to V_h^*$ and $B_h \colon V_h \to Q_h^*$ the operators associated with $a(\cdot,\cdot)\,|_{V_h \times V_h}$ and $b(\cdot,\cdot)\,|_{V_h \times Q_h}$. In contrast to the positive definite problem considered in (1.1), the existence and uniqueness of the discrete problem (1.8) does not follow from the result for the infinite dimensional problem (1.4). The reason is that in general the operator B_h does not correspond to the restriction of the operator B to V_h, i.e., $BV_h \not\subset Q_h^*$. Therefore, a proper balancing of the subspaces V_h and Q_h is required to ensure the existence and uniqueness of the solution of (1.8).

Theorem 1.5.2. *Existence and Uniqueness*

The saddle point problem (1.8) admits a solution $(u_h, p_h) \in V_h \times Q_h$, where $u_h \in V_h$ is uniquely determined and $p_h \in Q_h$ is unique up to an element of $Ker(B_h^*)$, if the following conditions hold for any $f \in V^*$ and $g \in Im(B)$:

- *The bilinear form $a(\cdot,\cdot)\,|_{V_h \times V_h}$ is $Ker(B_h)$-elliptic, i.e., there exists a constant $\alpha_h > 0$ such that*

$$a(v_{h0}, v_{h0}) \geq \alpha_h \|v_{h0}\|_V^2, \qquad v_{h0} \in Ker(B_h);$$

- *The bilinear form* $b(\cdot,\cdot)\,|_{V_h\times Q_h}$ *satisfies the Ladyzhenskaya-Babuška-Brezzi condition*

$$\inf_{q_h\in Q_h\backslash Ker(B_h^*)}\sup_{v_h\in V_h}\frac{b(v_h,q_h)}{\|v_h\|_{V_h}\|q_h\|_{Q_h\backslash Ker(B_h^*)}}\geq\beta_h>0.$$

The Ladyzhenskaya-Babuška-Brezzi condition is also known as discrete inf-sup condition.

1.5.3 Trace and Inverse Inequalities

For triangular tessellation, under the shape regularity assumption of the mesh, as given in Appendix A, we have the following lemmas; see (Wang & Ye, 2014) for details.

Lemma 1.5.3. *(Trace Inequality) There exists a constant* C *such that*

$$\|\theta\|_e^2\leq\left(h_T^{-1}\|\theta\|_T^2+h_T\|\nabla\theta\|_T^2\right),\qquad\forall T\in\mathcal{T}_h,\ e\in\mathcal{F}_h,$$

where $\theta\in H^1(T)$.

Lemma 1.5.4. *(Inverse Inequality) There exists a constant* $C=C(k)$ *such that*

$$\|\nabla\varphi\|_T\leq C(k)h_T^{-1}\|\varphi\|_T,\qquad\forall T\in\mathcal{T}_h$$

for any piecewise polynomial φ *of degree* k *on* $\mathcal{T}_h$.

These inequalities are critical to the desired approximation properties of the hybridizable discontinuous Galerkin (HDG) and weak Galerkin (WG) finite element methods, which will be the main focus of this study. And we will use them in our analysis of the BDDC operator designed for these methods.

1.6 Preconditioned Conjugate Gradient Method

Many practical engineering problems can be described by elliptic partial differential equations, the discretization of which by the finite element methods leads to large scale sparse

linear system. Direct solvers of such system usually become prohibitively expensive, if not impossible. A useful alternative is the iterative method based on Krylov subspaces.

Conjugate gradient (CG) method is a prototypical Krylov subspace method designed for symmetric positive definite system. But it is able to reach solution for a number of indefinite cases, although it may fail in general. The convergence rate of CG deteriorates with increasing condition number of the linear system. Thus, it is often desirable to design a suitable pre-conditioner for the original system so that preconditioned CG method can be applied, as it can be less computationally expensive, in terms of memory and computational time, compared with using other more robust iterative methods such as MINRES, BiCG, or GMRES (Šístek et al., 2011).

We furnish the relevant results of the CG methods here following (Toselli & Widlund, 2005). Let A be symmetric and positive definite. In the conjugate gradient method, the search directions p^k are chosen as *conjugate* with respect to A, i.e.,

$$\langle p^i, Ap^j \rangle = \langle p^i, p^j \rangle_A = 0, \quad i \neq j.$$

These vectors p^k are linear independent and provide a basis for the *Krylov spaces* $\mathcal{K}_k = \mathcal{K}_k(r_0, A) = span\{A^i r^0,\ i = 0, 1, \cdots, k-1\}$.

We write the algorithm as below:

Algorithm 1 Unpreconditioned conjugate gradient

1: Initialize: $r^0 = b - Au^0$
2: Iterate $k = 1, 2, \cdots$ until convergence
$\beta^k = \langle r^{k-1}, r^{k-1} \rangle / \langle r^{k-2}, r^{k-2} \rangle \qquad [\beta^1 = 0]$
$p^k = r^{k-1} + \beta^k p^{k-1} \qquad [p^1 = r^0]$
$\alpha^k = \langle r^{k-1}, r^{k-1} \rangle / \langle p^k, Ap^k \rangle$
$u^k = u^{k-1} + \alpha^k p^k$
$r^k = r^{k-1} - \alpha^k Ap^k$

Convergence of the unpreconditioned Conjugate Gradient depends on the condition number of A. We have the following result.

Lemma 1.6.1. *Let A be symmetric and positive definite. Then, the Conjugate Gradient method satisfies the error bound*

$$\left\|e^k\right\|_A \leq 2\eta_A^k \left\|e^0\right\|_A,$$

where the convergence factor is

$$\eta_A = \frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1},$$

where $\kappa(A)$, the condition number of A, is the ratio of the largest and smallest eigenvalues of A.

The conjugate gradient iteration provides an estimate of the eigenvalues of the matrix A (and thus of $\kappa(A)$). Let $R_k = [r_0/\|r_0\|, \cdots, r_{k-1}/\|r_{k-1}\|]$. One can prove that the restriction of A to $\mathcal{K}_k(r_0, A)$

$$T_k = R_k^T A R_k$$

is a symmetric, tridiagonal matrix, the entries of which can be constructed from the coefficients of the conjugate gradient iteration. By calculating the eigenvalues of T_k, one can easily obtain estimates of the largest and smallest eigenvalues of A.

When $\kappa(A)$ is large, preconditioning is necessary. Given a symmetric, positive definite matrix M, we can consider the modified linear system

$$M^{-1/2} A M^{-1/2} v = M^{-1/2} b, \quad v = M^{1/2} u.$$

Note that $M^{-1/2} A M^{-1/2}$ is symmetric and positive definite. We can then consider M as a *preconditioner* for A and apply Algorithm 1 to this modified system. After some manipulations, we have the following algorithm.

For the preconditioned system, we have the following result.

Lemma 1.6.2. *Let A and M be symmetric and positive definite. Then, the preconditioned*

Algorithm 2 Preconditioned conjugate gradient

1: Initialize: $r^0 = b - Au^0$
2: Iterate $k = 1, 2, \cdots$ until convergence
Precondition: $z^{k-1} = M^{-1} r^{k-1}$
$\beta^k = \langle z^{k-1}, r^{k-1} \rangle / \langle z^{k-2}, r^{k-2} \rangle \qquad [\beta^1 = 0]$
$p^k = z^{k-1} + \beta^k p^{k-1} \qquad [p^1 = z^0]$
$\alpha^k = \langle z^{k-1}, r^{k-1} \rangle / \langle p^k, Ap^k \rangle$
$u^k = u^{k-1} + \alpha^k p^k$
$r^k = r^{k-1} - \alpha^k A p^k$

Conjugate Gradient method satisfies the same error bound as in Lemma 1.6.1, with

$$\eta_A = \frac{\sqrt{\kappa(M^{-1}A)} - 1}{\sqrt{\kappa(M^{-1}A)} + 1}.$$

Eigenvalues of $M^{-1}A$ can also be obtained using the coefficients α^i and β^i in Algorithm 2.

1.7 Organization of the Dissertation

The remainder of this dissertation is organized as follows. We review the mixed and nonconforming finite elements methods relevant to this study in Chapter 2, and discuss the iterative substructuring methods in Chapter 3. In Chapter 4 and 5, we present BDDC algorithms for elliptic and Stokes problems, respectively. Both hybridizable discontinuous Galerkin and weak Galerkin methods are used to discretize the model problems. In Chapter 6, we draw conclusions from this study.

Chapter 2

Mixed and Non-conforming Finite Element Methods

Conforming methods are natural finite element methods in the sense that they approximate the function space underlying the variational formulation of the partial differential equations by its finite dimensional subspaces(Brezzi & Fortin, 1991). However, conforming elements are not always efficient, or even practical, for example, for partial differential equations with variable coefficients or over domain with curved boundaries(Tu, 2002). The nonconforming finite elements may become necessary in these cases. The term "non-conforming" refers to that the test functions do not form a subspace of the corresponding variational function space(Babuška & Zlámal, 1973).

In this section, we cast several methods relevant to this study within a single framework as applied to elliptic problems, aiming to reveal the connections among them. For the sake of simplicity, we restrict ourselves to the following model problem:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where Ω is polygonal domain and f a given function $\in L^2(\Omega)$.

To obtain the weak formulation, we introduce an auxiliary variable and rewrite the above

problem as follows:

$$q = -\nabla u, \quad \nabla \cdot q = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Let

$$V = H(div;\Omega) = \{v \in (L^2(\Omega))^n \mid \nabla \cdot v \in L^2(\Omega), n = 2,3\}, \tag{2.3}$$

$$W = L^2(\Omega). \tag{2.4}$$

Then in the mixed form, equation (2.2) is formulated as follows for the pair $(q,u) \in V \times W$:

$$(q,\tau) = (u, \nabla \cdot \tau), \quad \forall \tau \in V, \tag{2.5a}$$

$$(\nabla \cdot q, v) = (f,v), \quad \forall v \in W. \tag{2.5b}$$

Let $\mathcal{T}_h$ be a shape-regular tessellation of Ω with polygons in 2D and polyhedra in 3D. We denote the element in $\mathcal{T}_h$ by K, the diameter of K by h_K, and the area/volume of K by $|K|$. The mesh size is characterized by $h \coloneqq max_{K \in \mathcal{T}_h} h_K$. Define $\mathcal{F}_h$ be the set of edges/faces of elements $K \in \mathcal{T}_h$. $\mathcal{F}_h^i$ and $\mathcal{F}_h^\partial$ are subsets of $\mathcal{F}_h$, which consists of domain interior and boundary edges, respectively. We denote by $|e|$ the length/area of e and h_e the diameter of the edge/face in $\mathcal{F}_h$.

2.1 Mixed Finite Element

Some useful mixed finite element spaces introduced to approximate $H(div)$ include but not limited to, Raviart-Thomas(RT)(Raviart & Thomas, 1977) and Brezzi-Douglas-Marini (BDM)(Brezzi et al., 1985) elements. Here, we briefly introduce the RT elements for simpli-

cial triangulation $\mathcal{T}_h$ of Ω(Hoppe, 2016). For $K \in \mathcal{T}_h$, we set

$$R_k(\partial K) = \{\varphi \in L^2(\partial K) \mid \varphi \mid_e \in P_k(e),\ e \in \mathcal{F}_h(K)\},$$

where $P_k(e)$ is the set of polynomials defined on e with degree at most k.

Define

$$\boldsymbol{RT}_k(K) = (P_k(K))^n + \boldsymbol{x}\widetilde{P}_k(K), \quad k \geq 0, \tag{2.6}$$

where $\boldsymbol{x}$ is the position vector, $P_k(K)$ is the set of polynomials defined on K with degree at most k, n is the dimension of K, and $\widetilde{P}_k(K)$ is the set of homogeneous polynomials defined on K with degree at most k.

The Raviart-Thomas finite element space $\boldsymbol{RT}_k(\Omega; \mathcal{T}_h)$ is given by

$$\boldsymbol{RT}_k(\Omega; \mathcal{T}_h) = \{\boldsymbol{q} \in [L^2(\Omega)]^n \mid \boldsymbol{q} \mid_K \in \boldsymbol{RT}_k(K), K \in \mathcal{T}_h\}.$$

It is a finite dimensional subspace of $H(div; \Omega)$.

For $\boldsymbol{u} \in \boldsymbol{RT}_k(K)$, the degrees of freedom are given by

$$\int_{\partial K} \boldsymbol{q} \cdot \boldsymbol{n} p_k, \quad p_k \in R_k(\partial K),$$

$$\int_K \boldsymbol{q} \cdot \boldsymbol{p}_{k-1}, \quad \boldsymbol{p}_{k-1} \in [P_{k-1}(K)]^n.$$

We have

$$\dim \boldsymbol{RT}_k(K) = \begin{cases} (k+1)(k+3) & (n=2) \\ \frac{1}{2}(k+1)(k+2)(k+4) & (n=3) \end{cases}.$$

The mixed method leads to a saddle point problem to be solved. The finite element space pair $V_h \times W_h$ are finite dimensional subspaces of $H_0(div; \Omega) \times L_0^2(\Omega)$, and is subject to the inf-sup stability(Brezzi & Fortin, 1991).

The bilinear form can be written as follows: to find $(q_h, u_h) \in \boldsymbol{RT}_k(\Omega; \mathcal{T}_h) \times W_h$ such that:

$$(q_h, \tau) - \sum_{K \in \mathcal{T}_h} (u_h, \nabla \cdot \tau)_K = 0, \qquad \forall \tau \in \boldsymbol{RT}_k(\Omega; \mathcal{T}_h), \tag{2.7a}$$

$$\sum_{K \in \mathcal{T}_h} (\nabla \cdot q_h, v)_K = (f, v), \qquad \forall v \in W. \tag{2.7b}$$

2.2 Hybridized Mixed Finite Element

The standard mixed finite element spaces $V_h \times W_h \subset V \times W$ are finite dimensional and defined locally on each element. Denote $V_h \mid_K$ by $V_h(K)$ and $W_h \mid_K$ by $W_h(K)$. The constraint $V_h \subset V$ requires the normal component of the members of V_h to be continuous across the interior element boundaries $\mathcal{F}_h^i$(Chen, 1994). We relax this continuity constraint on V_h by defining the space

$$\widetilde{V}_h = \{v \in (L^2(\Omega))^n \mid v \mid_K \in V_h(K) \quad \text{for each} \quad K \in \mathcal{T}_h\}.$$

Then, we introduce Lagrange multipliers to enforce the required continuity on $\widetilde{V}_h$, and define

$$\mathcal{M}_h = \{\mu \in L^2(\mathcal{F}_h) \mid \mu \mid_e \in (V_h \cdot n) \mid_e \quad \text{for each} \quad e \in \mathcal{F}_h\},$$

where n is the outward normal direction of the edge. The hybrid form of the mixed method is to find $(q_h, u_h, \lambda_h) \in \widetilde{V}_h \times W_h \times \mathcal{M}_h$ such that

$$(q_h, \tau) - \sum_{K \in \mathcal{T}_h} (u_h, \nabla \cdot \tau)_K + \sum_{e \in \mathcal{F}_h^i} \langle \lambda_h, \tau \cdot n \rangle_e = 0, \qquad \forall \tau \in \widetilde{V}_h, \tag{2.8a}$$

$$\sum_{K \in \mathcal{T}_h} (\nabla \cdot q_h, v)_K = (f, v), \qquad \forall v \in W, \tag{2.8b}$$

$$\sum_{e \in \mathcal{F}_h^i} \langle q_h \cdot n, \mu \rangle_e = 0, \qquad \forall \mu \in \mathcal{M}_h. \tag{2.8c}$$

In fact, due to the continuity constraint (2.12c), the numerical solution $q_h \in V_h$. The equivalence between the hybridized mixed form and standard mixed form for certain finite element spaces can be proved following (Arnold & Brezzi, 1985; Brenner, 1992).

Equations (2.8) can be algebraically condensed to a symmetric positive definite system for λ_h(Chen, 1994).

2.3 Discontinuous Galerkin Methods

Discontinuous Galerkin(DG) method was first introduced by Reed and Hill for hyperbolic equations in 1973(Reed & Hill, 1973). Since then, this has been an active research area by both analysts and practitioners(Arnold et al., 2000). Independent development for the elliptic and parabolic equations using discontinuous finite elements was also proposed in the 1970's. These early works are generally referred to as interior penalty (IP) methods. It was later discovered that the DG methods bear remarkable similarities to the classical IP methods, and they can be cast in the same general framework.

If we multiply equation (2.2) by test function and integrate over each element K, we get

$$(q, \tau)_K = (u, \nabla \cdot \tau)_K - \langle u, n \cdot \tau \rangle_{\partial K}, \tag{2.9a}$$

$$-(q, \nabla v)_K + \langle q \cdot n, v \rangle_{\partial K} = (f, v)_K, \tag{2.9b}$$

where n is the outward normal unit vector to ∂K.

Set

$$\Sigma_h = \{\tau \in [L^2(\Omega)]^n \mid \tau \mid_K \in [P(K)]^n \quad \forall K \in \mathcal{T}_h\}.$$

$$V_h = \{v \in L^2(\Omega) \mid v \mid_K \in P(K) \quad \forall K \in \mathcal{T}_h\};$$

We consider the following general weak formulation: find $(q_h, u_h) \in \Sigma_h \times V_h$ such that, for each $K \in \mathcal{T}_h$, we have

$$(q_h, \tau)_K - (u_h, \nabla \cdot \tau)_K + \sum_{e \subset \partial K} \langle \widehat{u}_h^{e,K}, n \cdot \tau \rangle_e = 0 \quad \forall \tau \in \Sigma_h(K), \tag{2.10a}$$

$$-(q_h, \nabla v)_K + \sum_{e \subset \partial K} \langle \widehat{q}_h^{e,K} \cdot n, v \rangle_e = -(f, v)_K \quad \forall v \in V_h(K), \tag{2.10b}$$

where n is the outward normal unit vector to ∂K.

The test function spaces consist of generally piecewise continuous functions. For example, for triangular elements, the $V_h(K)$ can be the set of all polynomials of degree $p \leq 1$ and $\Sigma_h(K)$ can be all polynomial vector fields of degree $p-1$ or p. The constitutive relations defining the numerical fluxes (i.e.,$\widehat{u}^{e,K}$ and $\widehat{q}^{e,K}$) are crucial to the stability and accuracy of the method, as well as the sparsity and symmetry of the stiffness matrix (Arnold et al., 2000). Different choices will lead to different variant of the method.

2.4 Hybridizable Discontinuous Galerkin Methods

The interaction between ideas of DG and of the standard finite elements leads to the introduction of the Hybridizable Discontinuous Galerkin (HDG) Methods. The apparent advantage of the DG methods is their suitability for adaptation due to their flexibility with variable-degree polynomials and hanging nodes. The DG method was criticized for having too many degrees of freedoms and not easy to implement compared with the continuous Galerkin (CG) method; and for less accurate and less efficient in implementation compared with the mixed methods(Cockburn, 2010). It is in this historical context that HDG methods

were introduced for diffusion problems(Cockburn et al., 2009a).

The essence of HDG methods lies in the clever definition of the numerical trace so that the global coupled system can be reduced to edge variables only. Specifically, the numerical trace $\widehat{q}_h \cdot n$ is assumed to have the following form:

$$\widehat{q}_h \cdot n = q_h \cdot n + \tau(u_h - \widehat{u}_h) \quad \text{on } \partial K.$$

In this way, the function $\widehat{u}_h$ is determined by enforcing weakly the single-valuedness of the normal component of the numerical trace $\widehat{q}_h$ and by the Dirichlet boundary condition(Cockburn, 2010). Thus, for each edge $e \in \mathcal{F}_h$, we require that

$$\langle \mu, [\![\widehat{q} \cdot n]\!] \rangle_e = 0 \quad \forall \mu \in M(e), \tag{2.11a}$$

$$\widehat{u}_h = 0 \quad \text{if } e \in \mathcal{F}_h^{\partial}, \tag{2.11b}$$

where $[\![\widehat{q} \cdot n]\!] \coloneqq \widehat{q}^{+} \cdot n^{+} + \widehat{q}^{-} \cdot n^{-}$, and $M(e)$ is the space of approximate trace, which can be defined as a polynomial space of degree at most k with support on the edge e.

With this construction, we can determine (q_h, u_h) in terms of $(\widehat{u}_h, f)$ as the solution of the weak form (2.10). The discrete problem resulting from HDG discretization can be written as: to find $(q_h, u_h, \widehat{u}_h) \in \Sigma_h \times V_h \times M_h$ such that

$$(q_h, \tau) - \sum_{K \in \mathcal{T}_h} (u_h, \nabla \cdot \tau)_K + \sum_{e \in \mathcal{F}_h^i} \langle \widehat{u}_h, \tau \cdot n \rangle_e = 0, \qquad \forall \tau \in \Sigma_h, \tag{2.12a}$$

$$\sum_{K \in \mathcal{T}_h} -(q_h, \nabla v)_K + \sum_{e \in \mathcal{F}_h^i} \langle \widehat{q}_h \cdot n, v \rangle_e = (f, v), \qquad \forall v \in V_h, \tag{2.12b}$$

$$\sum_{e \in \mathcal{F}_h^i} \langle \widehat{q}_h \cdot n, \mu \rangle_e = 0, \qquad \forall \mu \in M_h. \tag{2.12c}$$

It can be proved that the HDG method is well defined if (i) $\tau > 0$ on $\mathcal{F}_h$, and if (ii) for

any element $K \in \mathcal{T}_h$, $\nabla w \in \Sigma_h(K)$ for all $w \in V_h(K)$; see Proposition 3.1. in (Cockburn, 2010).

2.5 Weak Galerkin Methods

The idea of weak Galerkin finite element scheme is to substitute the standard function and differential operators with the weakly defined counterparts. A weak function over the domain D is defined as $v = \{v_0, v_b\}$ such that $v_0 \in L^2(D)$ and $v_b \in H^{1/2}(\partial D)$. The v_0 part represents the value of v in the interior of D, while the v_b part represents the value of v on the boundary of D. Note that v_b does not bind itself with v_0 from the definition. In essence, weak functions relax the continuity property of the standard functions, thus to offer more flexibility in terms of variable representation. We denote by $\mathcal{W}(D)$ the space of weak functions over the domain D as below

$$\mathcal{W}(D) = \left\{ v = \{v_0, v_b\} : v_0 \in L^2(D), v_b \in H^{1/2}(\partial D) \right\},$$

and the corresponding vectorized weak function spaces by

$$[\mathcal{W}(D)]^n = \left\{ \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in \left[L^2(D)\right]^n, \mathbf{v}_b \in \left[H^{1/2}(\partial D)\right]^n \right\}.$$

The space of weak gradient or divergence operators will be defined as the dual space of appropriate Hilbert space, in similar manner as the dual of $L^2(D)$ can be identified with itself by using the L^2 inner product as the action of the linear functionals.

Definition 2.5.1. *For any $v \in \mathcal{W}(D)$, the weak gradient of v is defined as the linear functional $\nabla_w v$ in the dual space of $H(div; D)$ whose action on each $\mathbf{q} \in H(div; D)$ is given by*

$$(\nabla_w v, \mathbf{q})_D = -(v_0, \nabla \cdot \mathbf{q})_D + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial D},$$

where $\mathbf{n}$ is the outward normal direction to $\partial\Omega$.

Definition 2.5.2. *For any* $\mathbf{v} \in [W(D)]^n$*, the weak divergence of* $\mathbf{v}$ *is defined as the linear functional* $\nabla_w \cdot \mathbf{v}$ *in the dual space of* $H^1(D)$ *whose action on each* $\varphi \in H^1(D)$ *is given by*

$$(\nabla_w \cdot \mathbf{v}, \varphi)_D = -(\mathbf{v}_0, \nabla\varphi)_D + \langle \mathbf{v}_b \cdot \mathbf{n}, \varphi \rangle_{\partial D},$$

where $\mathbf{n}$ *is the outward normal direction to* $\partial\Omega$*.*

Now, we are in a position to introduce the weak Galerkin finite element algorithm. For any domain D, let $P_k(D)$ be the space of polynomials of degree $\leq k$ on D. Define the weak Galerkin finite element spaces for the velocity variable associated with $\mathcal{T}_h$ as follows:

$$V_k = \left\{ v = \{v_0, v_b\} : \{v_0, v_b\}|_K \in [P_k(K)]^n \times [P_{k-1}(e)]^n, \forall K \in \mathcal{T}_h, e \subset \partial K \right\}.$$

Note that a function $v \in V_k$ has a single value v_b on each edge $e \in \mathcal{F}_h$. The subspace of V_k with vanishing boundary values on $\partial\Omega$ is denoted by

$$V_k^0 = \{ v = \{v_0, v_b\} \in V_k : v_b = 0 \text{ on } \partial\Omega \}.$$

For the pressure variable, define the following finite element space

$$W_{k-1} = \left\{ q : q \in L_0^2(\Omega), q|_K \in P_{k-1}(K) \right\}.$$

Denote the discrete weak gradient operator by $\nabla_{w,k-1}$, and the discrete weak divergence operator by $\left(\nabla_{w,k-1}\cdot\right)$, respectively. They are defined on the finite element space V_k as follows: for $v = \{v_0, v_b\} \in V_k$, on each element $K \in \mathcal{T}_h$, $\nabla_{w,k-1} v|_K \in [P_{k-1}(K)]^n$ and $\nabla_{w,k-1} \cdot v|_K \in P_{k-1}(K)$ are the unique solutions of the following equations, respectively,

$$\left(\nabla_{w,k-1} v|_K, \mathbf{q}\right)_K = -\left(v_{0,K}, \nabla\cdot\mathbf{q}\right)_K + \left\langle v_{b,K}, \mathbf{q}\cdot\mathbf{n}\right\rangle_{\partial K}, \qquad \forall \mathbf{q} \in [P_{k-1}(K)]^n,$$

$$\left(\nabla_{w,k-1} \cdot v|_K, \varphi\right)_K = -\left(v_{0,K}, \nabla\varphi\right)_K + \left\langle v_{b,K}\cdot\mathbf{n}, \varphi\right\rangle_{\partial K}, \qquad \forall \varphi \in P_{k-1}(K),$$

where $v_{0,K}$ and $v_{b,K}$ are the restrictions of v_0 and v_b to K, respectively, $(u,w)_K = \int_K uwdx$, and $\langle u,w\rangle_{\partial K} = \int_{\partial K} uwds$. To simplify the notation, we shall drop the subscript $k-1$ in the notation $\nabla_{w,k-1}$ and $\left(\nabla_{w,k-1}\cdot\right)$ for the discrete weak gradient and the discrete weak divergence operator. We denote the L^2 inner product over the triangulation as a summation over each element of the triangulation, for example,$(\nabla_w u, \nabla_w w)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}(\nabla_w u, \nabla_w w)_K$, $(\nabla_w \cdot v, q)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}(\nabla_w \cdot v, q)_K$.

Denote by Q_0 the L^2 projection from $L^2(K)$ onto $P_k(K)$, and denote by Q_b the L^2 projection from $L^2(e)$ onto $P_{k-1}(e)$, for $e \in \mathcal{F}_h$. And we write the correponding projection operator for the weak function as $Q_h = \{Q_0, Q_b\}$.

The discrete problem resulting from the WG discretization can then be written as: find $u_h = \{u_0, u_b\} \in V_k^0$ such that

$$(\nabla_w u_h, \nabla_w v_h)_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h} h_K^{-1}\langle Q_b u_0 - u_b, Q_b v_0 - v_b\rangle_{\partial K} = (f, v_0), \forall v = \{v_0, v_b\} \in V_k^0. \quad (2.13)$$

In this study, we will focus on HDG and WG methods. We used the Lagrange triangle in the simulation. The nodal basis functions for the linear and quadratic elements are given in Appendix B.

Chapter 3

Iterative Substructuring Methods

3.1 Introduction

The basic idea of domain decomposition is quite natural and simple, as the name suggests itself. Based on the partition of the domain on which the original problem is defined, it can be categorized as overlapping domain decomposition and non-overlapping domain decomposition method. The latter is also called iterative substructuring methods, and this is an important family of domain decomposition methods. We can think of this class of methods as eliminating the interior variables of elements to some stage and solving the reduced linear system by a preconditioned Krylov subspace method(Toselli & Widlund, 2005). An iterative sub-structuring method can be further classified as either of primal type or of dual type. It is of primal type if the reduced linear system is given in terms of a subset of the original finite element degrees of freedom on the union of the interfaces between the substructures; it is of dual type if the principal unknowns of the iteration are Lagrange multipliers which enforce the continuity constraints of the solution across subdomain interfaces(Toselli & Widlund, 2005). The following definitions (Toselli & Widlund, 2005) are cited, as they are frequently used to measure the performance of BDDC algorithms.

Definition 3.1.1 (Optimality)**.** *An iterative method for the solution of a linear system is*

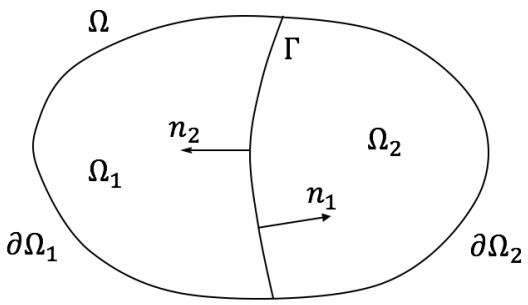

Figure 3.1: Partition into two non-overlapping subdomains.

said to be optimal, if its rate of convergence to the exact solution is independent of the size of the system.

Definition 3.1.2 (Scalability)**.** *A domain decomposition iterative method for the solution of a linear system is said to be scalable, if its rate of convergence does not deteriorate when the number of subdomains grows. This typically means that convergence does not deteriorate when H, the typical subdomain size, becomes small.*

3.1.1 Problem Setting

We consider a domain Ω subdivided into two non-overlapping subdomains Ω_1 and Ω_2. The interface in between is denoted by Γ. A schematic diagram is illustrated in Fig. 3.1.

Consider a finite element approximation of a Poisson problem on Ω.

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

Set up a load vector and a stiffness matrix for each subdomain

$$f^{(i)} = \begin{bmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{bmatrix}, \quad A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}, \quad i = 1, 2.$$

In practice, we also need to decompose the whole domain into multi-subdomains. We

decompose Ω into N non overlapping subdomain Ω_i with diameters H_i, $i = 1,\ldots,N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such elements forming an individual subdomain is uniformly bounded. We define edges/faces as open sets shared by two subdomains. Two nodes belong to the same face when they are associated with the same pair of subdomains. Let Γ be the interface between the subdomains. The set of the interface nodes $\mathit{\Gamma}_h$ is defined as $\mathit{\Gamma}_h := \left(\cup_{i\neq j}\partial\Omega_{i,h}\cap\partial\Omega_{j,h}\right)\setminus\partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is that of $\partial\Omega$. We assume the triangulation of each subdomain is quasi-uniform.

Let $V^{(i)}$ be the finite element space over Ω_i. Each $V^{(i)}$ can be decomposed into a subdomain interior part $V_I^{(i)}$ and a subdomain interface part $V_\Gamma^{(i)}$. The subdomain interface part $V_\Gamma^{(i)}$ can be further decomposed into a primal subspace $V_\Pi^{(i)}$ and a dual subspace $V_\Delta^{(i)}$. Namely, we have

$$V^{(i)} = V_I^{(i)}\bigoplus V_\Gamma^{(i)} = V_I^{(i)}\bigoplus V_\Pi^{(i)}\bigoplus V_\Delta^{(i)}.$$

The corresponding spaces over the domain Ω will be

$$V = V_I\bigoplus V_\Gamma = V_I\bigoplus V_\Pi\bigoplus V_\Delta,$$

with $V = \prod_{i=1}^N V^{(i)}$, $V_I = \prod_{i=1}^N V_I^{(i)}$, $V_\Gamma = \prod_{i=1}^N V_\Gamma^{(i)}$, $V_\Pi = \prod_{i=1}^N V_\Pi^{(i)}$, and $V_\Delta = \prod_{i=1}^N V_\Delta^{(i)}$. In general, the functions in the space V_Γ are discontinuous across the interface. The standard finite element space are continuous across the interface, and we denote this continuous subspace of V_Γ by $\widehat{V}_\Gamma$. An intermediate space, which is continuous at the primal variables and generally discontinuous at the dual variables, is referred to as $\widetilde{V}_\Gamma$.

We introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)} : \widetilde{V}_\Gamma \to V_\Gamma^{(i)}$ restricts functions in the space $\widetilde{V}_\Gamma$ to the components $V_\Gamma^{(i)}$ of the subdomain Ω_i. $\overline{R}_\Gamma : \widetilde{V}_\Gamma \to V_\Gamma$ is the direct sum of $\overline{R}_\Gamma^{(i)}$. $R_\Delta^{(i)} : \widehat{V}_\Gamma \to V_\Delta^{(i)}$ maps the functions from $\widehat{V}_\Gamma$ to $V_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi} : \widehat{V}_\Gamma \to \widehat{V}_\Pi$ is a restriction operator from $\widehat{V}_\Gamma$ to its subspace $\widehat{V}_\Pi$. $\widetilde{R}_\Gamma : \widehat{V}_\Gamma \to \widetilde{V}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define the positive

scaling factor $\delta_i^\dagger(x)$ as follows:

$$\delta_i^\dagger(x) = \frac{1}{card(\mathcal{I}_x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{I}_x$ is the set of indices of the subdomains that have x on their boundaries, and $card(\mathcal{I}_x)$ counts the number of the subdomain boundaries to which x belongs. It is clear that $\delta_i^\dagger(x)$'s provide a partition of unity, i.e., $\sum_{i\in\mathcal{I}_x}\delta_i^\dagger(x) = 1$, for any $x \in \Gamma_h$. We note that $\delta_i^\dagger(x)$ is constant on each edge. Multiplying each row of $R_\Delta^{(i)}$ with the scaling factor gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{D,\Delta}^{(i)}$.

3.1.2 Schur Complement Systems

If we use exact solvers for the subdomain problems, we can often reduce our discussion to one about Schur complement(Widlund, 2011). Schur complement with respect to unknowns at interface Γ is defined as

$$S^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} {A_{II}^{(i)}}^{-1} A_{I\Gamma}^{(i)}. \tag{3.2}$$

The corresponding condensed load vector is

$$g^{(i)} = f^{(i)} - A_{\Gamma I}^{(i)} {A_{II}^{(i)}}^{-1} f^{(i)}. \tag{3.3}$$

In practice, we don't form the Schur complement explicitly, but realize its action by block-Gaussian elimination. The reduced subdomain interface problem, obtained by assembling the Schur complement from each subdomain, has a reduction in dimension of the Krylov space vectors, and, even better, the condition number of this reduced system will be smaller than the original linear system for a symmetric positive definite problem. Therefore, using iterative substructuring method, the Schur complement system usually converges much faster than the original system. Once the interface values are obtained, we can calculate the values in the interiors by solving a Dirichlet problem for each subdomain(Widlund, 2011).

3.1.3 Discrete Harmonic Extensions

Let $A^{(i)}$ be the discrete form of the harmonic operator. A function u is defined to be *discrete harmonic* if it is the solution of the linear system $A^{(i)}$ with a right-hand side of the form $(0, f_\Gamma^T)^T$. The second part of the solution equals to ${S^{(i)}}^{-1} f_\Gamma$(Widlund, 2011). This u is $A^{(i)}-$orthogonal to any v which vanishes on Γ, and provides the *minimal energy extension* for given values on Γ. Therefore, the discrete harmonic function space has direct correspondence with the Schur complement and the node values at the interface Γ. Specifically, we have

$$A_{II}^{(i)} u_I^{(i)} + A_{I\Gamma}^{(i)} u_\Gamma^{(i)} = 0.$$

In what follows, we use $\mathcal{H}_i\left(u_\Gamma^{(i)}\right)$ to denote the discrete harmonic extension operator over the domain Ω_i, and $\mathcal{H}(u_\Gamma)$ to denote the piecewise discrete harmonic extension operator over the domain Ω.

For completeness of discussion, we cite the following lemmas to establish the equivalence between discrete harmonic extension over the domain Ω and the trace function defined over the relevant domain interface boundaries $\partial\Omega_i \cap \Gamma$ (Toselli & Widlund, 2005; Tu, 2006).

Lemma 3.1.1. *Let $u_\Gamma^{(i)}$ be the restriction of a finite element function to $\partial\Omega_i \cap \Gamma$. Then, the discrete harmonic extension $u^{(i)} = \mathcal{H}_i\left(u_\Gamma^{(i)}\right)$ of $u_\Gamma^{(i)}$ into Ω_i satisfies*

$$a_i\left(u^{(i)}, u^{(i)}\right) = \min_{v^{(i)}|_{\partial\Omega_i \cap \Gamma} = u_\Gamma^{(i)}} a_i\left(v^{(i)}, v^{(i)}\right)$$

and

$$u_\Gamma^{(i)\mathsf{T}} S^{(i)} u_\Gamma^{(i)} = a_i\left(u^{(i)}, u^{(i)}\right)$$

Analogously, if u_Γ is the restriction of a finite element function to Γ, the piecewise discrete harmonic extension $u = \mathcal{H}(u_\Gamma)$ of u_Γ into the interior of the subdomains satisfies

$$a(u, u) = \min_{v|_\Gamma = u_\Gamma} a(v, v)$$

and

$$u_\Gamma{}^\mathsf{T} S u_\Gamma = a(u,u).$$

This lemma says that we can work with functions defined on the interface Γ and the corresponding discrete harmonic extension interchangeably. The next lemma will establish the equivalence of norms between these two types of functions.

Lemma 3.1.2. *For floating subdomains, let u be discrete harmonic. Then, there exist positive constants c and C, independent of h and H, such that*

$$c\|u_\Gamma\|^2_{H^{1/2}(\partial\Omega_i\cap\Gamma)} \leq \|u\|^2_{H^1(\Omega_i)} \leq C\|u_\Gamma\|^2_{H^{1/2}(\partial\Omega_i\cap\Gamma)},$$

$$c|u_\Gamma|^2_{H^{1/2}(\partial\Omega_i\cap\Gamma)} \leq |u|^2_{H^1(\Omega_i)} \leq C|u_\Gamma|^2_{H^{1/2}(\partial\Omega_i\cap\Gamma)}.$$

Consequently,

$$c\rho_i|u_\Gamma|^2_{H^{1/2}(\partial\Omega_i\cap\Gamma)} \leq u_\Gamma^{(i)\mathsf{T}} S^{(i)} u_\Gamma^{(i)} \leq C\rho_i|u_\Gamma|^2_{H^{1/2}(\partial\Omega_i\cap\Gamma)}$$

with $u_\Gamma^{(i)}$ the restriction of u to $\partial\Omega_i\cap\Gamma$ and the constants independent of h, H, and the ρ_i.

For subdomains intersecting with the boundary, i.e., $\partial\Omega_i\cap\partial\Omega$ has a non-vanishing measure, we have similar results

$$c\|u_\Gamma\|^2_{H^{1/2}_{00}(\partial\Omega_i\cap\Gamma)} \leq \|u\|^2_{H^1(\Omega_i)} \leq C\|u_\Gamma\|^2_{H^{1/2}_{00}(\partial\Omega_i\cap\Gamma)},$$

Further by Friedrichs inequality, the H^1 seminorm is equivalent to the H^1 norm for functions defined over this type of subdomains. Thus, the result for the seminorm follows

$$c|u_\Gamma|^2_{H^{1/2}_{00}(\partial\Omega_i\cap\Gamma)} \leq |u|^2_{H^1(\Omega_i)} \leq C|u_\Gamma|^2_{H^{1/2}_{00}(\partial\Omega_i\cap\Gamma)}.$$

Consequently,

$$c\rho_i\|u_\Gamma\|^2_{H^{1/2}_{00}(\partial\Omega_i\cap\Gamma)} \leq u_\Gamma^{(i)\mathsf{T}} S^{(i)} u_\Gamma^{(i)} \leq C\rho_i\|u_\Gamma\|^2_{H^{1/2}_{00}(\partial\Omega_i\cap\Gamma)}.$$

Next, we will largely follow the developments in (Toselli & Widlund, 2005; Li & Widlund, 2006b; Tu, 2006) to introduce the BBDC algorithm and its earlier versions.

3.2 The Neumann-Neumann Methods

For simplicity, we start from the two subdomain case. In terms of differential operator, the basis Neumann-Neumann algorithm can be written as follows, for $n > 0$:

$$
\begin{aligned}
(D_i) &\left\{ \begin{array}{rcll} -\Delta u_i^{n+1/2} &=& f \quad \text{in} \quad \Omega, \\ u_i^{n+1/2} &=& 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma, \\ u_i^{n+1/2} &=& u_\Gamma^n \quad \text{on} \quad \Gamma \end{array} \right\}, \quad i = 1, 2, \\
(N_i) &\left\{ \begin{array}{rcll} -\Delta \psi_i^{n+1} &=& 0 & \text{in} \quad \Omega, \\ \psi_i^{n+1} &=& 0 & \text{on} \quad \partial\Omega \setminus \Gamma, \\ \frac{\partial \psi^i}{\partial n_i}^{n+1} &=& \frac{\partial u_1}{\partial n_1}^{n+1/2} + \frac{\partial u_2}{\partial n_2}^{n+1/2} & \text{on} \quad \Gamma, \end{array} \right\}, \quad i = 1, 2, \qquad (3.4) \\
& u_\Gamma^{n+1} = u_\Gamma^n - \theta(\psi_1^{n+1} + \psi_2^{n+1}) \quad \text{on} \quad \Gamma,
\end{aligned}
$$

with a suitable $\theta \in (0, \theta_{max})$. If we define the vectors of the interior degrees of freedom as $v_i = u_I^{(i)}$ and $w_i = \psi_I^{(i)}$, we have the following matrix form

$$
\begin{aligned}
& (D_i)\ A_{II}^{(i)} v_i^{n+1/2} + A_{I\Gamma}^{(i)} u_\Gamma^n = f_I^{(i)}, \quad i = 1, 2, \\
& (N_i) \begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} w_i^{n+1} \\ \eta_i^{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ r_\Gamma \end{bmatrix}, \quad i = 1, 2, \qquad (3.5) \\
& u_\Gamma^{n+1} = u_\Gamma^n - \theta(\eta_1^{n+1} + \eta_2^{n+1}),
\end{aligned}
$$

where the residual r_Γ is defined as

$$r_\Gamma = (A_{\Gamma I}^{(1)} v_1^{n+1/2} + A_{\Gamma\Gamma}^{(1)} u_\Gamma^n - f_\Gamma^{(1)}) + (A_{\Gamma I}^{(2)} v_2^{n+1/2} + A_{\Gamma\Gamma}^{(2)} u_\Gamma^n - f_\Gamma^{(2)}).$$

We eliminate $v_i^{n+1/2}$ and w_i^{n+1} from (3.5). The preconditioned matrix of this system can be written as

$$FS = ({S^{(1)}}^{-1} + {S^{(2)}}^{-1})S = ({S^{(1)}}^{-1} + {S^{(2)}}^{-1})(S^{(1)} + S^{(2)}),$$

the action of which to a vector involves the solution of two Dirichlet problems and two problems with Neumann data on Γ. For a full development, refer to (Toselli & Widlund, 2005, Chapter 1).

This algorithm can be easily extended to multi-subdomains. In this case, we solve the interface Schur complement problem: find $u_\Gamma \in \widehat{V}_\Gamma$, such that

$$\widehat{S}_\Gamma u_\Gamma = \sum_{i=1}^{N} {R_\Gamma^{(i)}}^T g_\Gamma^{(i)}, \tag{3.6}$$

where $g_\Gamma^{(i)}$ is the subdomain interface load vector as defined in (3.3), and $\widehat{S}_\Gamma$ is the interface Schur complement operator defined on the space $\widehat{V}_\Gamma$. In particular, $\widehat{S}_\Gamma$ can be represented as $\widehat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^{N} ({R_\Gamma^{(i)}}^T S^{(i)} R_\Gamma^{(i)})$. The one-level Neumann-Neumann preconditioner can be written as (Li & Widlund, 2006b)

$$M_{NN}^{-1} = \sum_{i=1}^{N} {R_{D,\Gamma}^{(i)}}^T {S^{(i)}}^\dagger R_{D,\Gamma}^{(i)}. \tag{3.7}$$

The disadvantages of this algorithm is that it needs to deal with singular subdomain Schur complement, and that the convergence rate will deteriorate with increasing number of subdomains. See (Li & Widlund, 2006b) for details.

3.3 Balancing Neumann-Neumann Methods

Two-level balancing Neumann-Neumann method has been proposed to improve the convergence performance of the one-level methods. This type of method introduces a coarse space V_0 to the algorithm. For example, we can choose a minimal coarse space as

$$V_0 = \text{span}\{R_i^T \delta_i^\dagger,\ \partial\Omega_i \cap \partial\Omega = \emptyset\}.$$

The balancing Neumann-Neumann preconditioner applicable to the interface system (3.6) can be written as follows:

$$M_{BNN}^{-1} = R_0^T S_0^{-1} R_0 + (I - R_0^T S_0^{-1} R_0 \widehat{S}_\Gamma)(\sum_{i=1}^{N} {R_{D,\Gamma}^{(i)}}^T {S^{(i)}}^\dagger R_{D,\Gamma}^{(i)})(I - \widehat{S}_\Gamma R_0^T S_0^{-1} R_0),$$

where $S_0 = R_0 \widehat{S}_\Gamma R_0^T$.

This preconditioner can be recast in the abstract Schwarz framework. It is a hybrid Schwarz method with a minimal coarse space designed to make all local Neumann problems solvable. The convergence rate of these algorithms are independent of the number of subdomains, and poly-logarithmically dependent on the subdomain problem size. A considerable improvement was later made to this method, and termed as the Balancing Domain Decomposition by Constraints (BDDC) methods(Dohrmann, 2003), as will be introduced below.

3.4 BDDC Methods

The main idea of the BDDC pre-conditioner is to construct a partially assembled finite element space $\widetilde{V}_\Gamma$ such that

$$\widehat{V}_\Gamma \subset \widetilde{V}_\Gamma \subset V_\Gamma.$$

We can define a partially assembled Schur complement $\widetilde{S}_\Gamma$ on $\widetilde{V}_\Gamma$, and obtain the fully

assembled Schur complement $\widehat{S}_\Gamma$ by $\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma$. Therefore, the reduced interface problem can be written as: find $u_\Gamma \in \widehat{V}_\Gamma$ such that

$$\widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_\Gamma = g_\Gamma.$$

The preconditioned BDDC method is of the form

$$M_{BDDC}^{-1} \widehat{S}_\Gamma u_\Gamma = M_{BDDC}^{-1} g_\Gamma,$$

where the preconditioner is defined as follows

$$M_{BDDC}^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}, \tag{3.8}$$

with

$$\widetilde{S}_\Gamma^{-1} = R_{\Gamma\Delta}^T \left(\sum_{i=1}^{N} \begin{bmatrix} 0 & R_\Delta^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\triangle I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R_\Delta^{(i)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi S_\Pi^{-1} \Phi^T \tag{3.9a}$$

$$\Phi = R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T \left(\sum_{i=1}^{N} \begin{bmatrix} 0 & R_\Delta^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\triangle I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} R_\Pi^{(i)} \right) \tag{3.9b}$$

$$S_\Pi = \sum_{i=1}^{N} R_\Pi^{(i)T} \left(A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\triangle I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} \right) R_\Pi^{(i)} \tag{3.9c}$$

In the BDDC preconditioner, the coarse problem is proposed across the interface formed by parts of the boundaries of at least two subdomains to enforce the continuity constraints at the primal variables. One advantage with such designed coarse problem is that the resultant Schur complements are invertible. Similar poly-logarithmic condition number estimate as for the balancing Neumann-Neumann methods can be achieved. For more detailed discussions, see (Toselli & Widlund, 2005).

Chapter 4

BDDC Algorithms for Elliptic Problem

The main results from the first section were published in (Tu & Wang, 2016); those from the second section were published in (Tu & Wang, 2017c).

4.1 BDDC for Elliptic Problem with HDG Method

4.1.1 Introduction

In this work, a Balancing Domain Decomposition by Constraints (BDDC) algorithm is developed for the hybridizable discontinuous Galerkin (HDG) method. General HDG methods were introduced by Cockburn and his collaborators in (Cockburn et al., 2009a) and the specific HDG method we consider here is often called LDG-H method, which is constructed by using the local discontinuous Galerkin method on each element. One distinct feature of the HDG method is that the only global coupled degrees of freedom are a scalar variable, called "numerical traces". Therefore the resulting global system from the HDG is much smaller than other traditional DG methods. The superconvergence of HDG methods have also been studied in (Cockburn et al., 2008, 2009b). Recently, in (Cockburn et al., 2014), the condition

number of the linear system, arising from the HDG (LDG-H) discretization of a second order elliptic problem, has been shown to grow like $O(h^{-2})$ if $\tau h \leq C$. Here τ is the typical penalty constant, h is the typical mesh size, and C is a constant. For so-called "super-penalized" cases where τ is chosen to be $O(\frac{1}{h^{\alpha}})$ with $\alpha > 1$, the condition number grows even faster. Therefore efficient fast solvers for the linear system are necessary.

There are many fast solvers for DG methods and their variants such as multigrid and domain decomposition methods. Geometric Multigrid methods for the interior penalty DG were studied in (Gopalakrishnan & Kanschat, 2003b) and extended to other DG methods in (Gopalakrishnan & Kanschat, 2003a) using the unified analysis of (Arnold et al., 2002). Algebraic multigrid methods have been studied in (Kraus & Tomar, 2008a,b). In (Feng & Karakashian, 2001, 2005), two-level additive Schwarz methods were developed for second order elliptic problems and two-level non-overlapping Schwarz methods were studied for four order biharmonic equations, respectively. Overlapping Schwarz preconditioners were developed for advection-diffusion problems in (Lasser & Toselli, 2003). In (Antonietti & Ayuso, 2007, 2008, 2009; Ayuso de Dios & Zikatanov, 2009), a class of of Schwarz preconditioners were studied for different problems. Several nonoverlapping domain decomposition methods are developed, in (Dryja et al., 2007, 2012, 2013), for the discretization using a conforming finite element inside each subdomain and a discontinuous Galerkin method across subdomain boundary. An overlapping Schwarz and a nonoverlapping (BDDC) domain decomposition methods are studied in (Barker et al., 2011; Brenner et al., 2013) for a weakly over-penalized symmetric interior penalty method. Similar algorithms have been developed for a class of staggered discontinuous Galerkin methods in (Chung et al., 2013; Kim et al., 2014). A BDDC algorithm is studied for more general DG methods in (Diosady & Darmofal, 2012) based on the unified analysis of (Arnold et al., 2002).

However, there are relatively fewer fast solvers for the HDG methods. A multigrid V-cycle was used as a linear solver for the HDG in (Cockburn et al., 2014). Both overlapping and nonoverlapping domain decomposition methods are studied for high order HDG method

in (Schöberl & Lehrenfeld, 2013), where the domain decomposition algorithms are applied on the element level (namely one element is considered as a subdomain).

The BDDC algorithms, introduced by Dohrmann for second order elliptic problem in (Dohrmann, 2003), see also (Mandel & Dohrmann, 2003; Mandel et al., 2005), are nonoverlapping domain decomposition methods, which are similar to the balancing Neumann-Neumann (BNN) algorithms. In BDDC, the coarse problems are given in terms of a set of primal constraints. An important advantage with such a coarse problem is that the Schur complements that arise in the computation will all be invertible. The BDDC algorithms have been extended to the second order elliptic problem with mixed and hybrid formulations in (Tu, 2005, 2007a) and the Stokes problem (Li & Widlund, 2006a).

In this work, we consider the BDDC algorithm for the linear system arising from the HDG method. The close relationship between HDG and the classical hybrizied Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) methods was highlighted in (Cockburn et al., 2009a). In (Cockburn et al., 2008), it has been shown that a specific HDG method has an exactly same stiffness matrix as the hybridized RT and BDM methods. In (Cockburn et al., 2014), an important spectral relation between the bilinear form resulting from the HDG and hybridized RT method is established. As a result, the previous developed preconditioners for the hybrid RT methods can be applied to the HDG, such as the overlapping Schwarz preconditioner in (Gopalakrishnan, 2003), multigrid preconditioner in (Gopalakrishnan & Tan, 2009), and the BDDC preconditioner in (Tu, 2007a). Here, we apply the BDDC preconditioner directly to the HDG bilinear form and estimate the condition number bound of the resulting preconditioned operator using its spectral relation with hybridized RT method. Compared to the multigrid algorithms studied in (Cockburn et al., 2014), the BDDC algorithm is applied directly to the system arising from the HDG method. In (Tu, 2007a), only the lowest order Raviart-Thomas finite element method is considered. Here, in our analysis, we also include high order elements. For the dependence of the condition number bound on the order of the element, we need to examine such dependence in sev-

eral norms including those derived from the bilinear forms of the HDG and hybridized RT methods. Refined analysis of the condition number bound is needed for this dependence and will be given in future study. For some results related to this issue, see (Bică, 1997, 1998; Schöberl & Lehrenfeld, 2013).

4.1.2 An elliptic problem and HDG discretization

We consider the following elliptic problem on a bounded polygonal domain Ω, in two/three dimensions, with a Dirichlet boundary condition:

$$\begin{cases} -\nabla \cdot (a\nabla u) = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where a is a positive definite matrix function with the entries in $L^\infty(\Omega)$ satisfying

$$\xi^T a(\boldsymbol{x})\xi \geq \alpha \|\xi\|^2, \quad \text{for a.e.} \quad \boldsymbol{x} \in \Omega, \tag{4.2}$$

for some positive constant α. $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Omega)$. Without loss of generality, we assume that $g = 0$. The equation (4.1) has a unique solution $u \in H^2(\Omega)$; see (Braess, 2007).

We then introduce a new variable $\boldsymbol{q}$:

$$\mathbf{q} = a\nabla u. \tag{4.3}$$

and let $\rho = a^{-1}$. We obtain the following system for $\mathbf{q}$ and u as

$$\begin{cases} -\rho\mathbf{q} = -\nabla u & \text{in} \quad \Omega, \\ -\nabla \cdot \mathbf{q} = f & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{4.4}$$

We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$,

and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such triangles forming an individual subdomain is uniformly bounded. We also assume $a(\boldsymbol{x})$, the coefficient of (4.1), is constant in each subdomain. Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω with charateristic element size h and the element in $\mathcal{T}_h$ denoted by κ. Define $\mathcal{E}$ to be the union of edges of elements κ. $\mathcal{E}^i$ and $\mathcal{E}^\partial$ are the sets of the domain interior and boundary edges, respectively.

Let $P^k(D)$ be the space pf polynomials of order at most k on D and We set $\mathbf{P}^k(D) = [P^k(D)]^2$ and define the following finite element spaces:

$$\begin{aligned}
\mathbf{V}^k &= \{\mathbf{v}_h \in [L^2(\Omega)]^2 : \mathbf{v}_h|_\kappa \in \mathbf{P}^k(\kappa) \quad \forall \kappa \in \Omega\},\\
W^k &= \{w_h \in L^2(\Omega) : w_h|_\kappa \in P^k(\kappa) \quad \forall \kappa \in \Omega\},\\
M^k &= \{\mu_h \in L^2(\mathcal{E}) : \mu_h|_e \in P^k(e) \quad \forall e \in \mathcal{E}\}.
\end{aligned}$$

Let $\Lambda^k = \{\mu \in M^k : \mu = 0 \text{ on } \partial\Omega\}$. To make our notations simple, we drop the superscript k from now on.

For each κ, we find $(\mathbf{q}_h, u_h) \in (\mathbf{V}(\kappa), W(\kappa))$ such that for all $\kappa \in \mathcal{T}_h$

$$\begin{cases}
-(\rho\mathbf{q}_h, \mathbf{v}_h)_\kappa - (u_h, \nabla\cdot\mathbf{v}_h)_\kappa + \langle\widehat{\mathbf{u}}_h, \mathbf{v}_h\cdot\mathbf{n}\rangle_{\partial\kappa} &=& 0 & \forall \mathbf{v}_h \in \mathbf{V}(\kappa),\\
(\mathbf{q}_h, \nabla w_h)_\kappa - \langle\widehat{\mathbf{q}}_h\cdot\mathbf{n}, w_h\rangle_{\partial\kappa} &=& (f, w_h)_\kappa & \forall w_h \in W(\kappa),
\end{cases} \tag{4.5}$$

where $(\cdot,\cdot) = \int_\kappa$ and $\langle\cdot,\cdot\rangle_{\partial\kappa} = \int_{\partial\kappa}$. $\widehat{\mathbf{u}}_h$ and $\widehat{\mathbf{q}}_h$ are the numerical traces which approximate u_h and $\mathbf{q}_h$ on $\partial\kappa$ respectively.

Let $\lambda_h \in \Lambda$ and the numerical trace $\widehat{\mathbf{u}}_h = \lambda_h$. The numerical flux $\widehat{\mathbf{q}}\cdot\mathbf{n}$ is more complicated and takes the form:

$$\widehat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \lambda_h), \quad \text{on } \partial\kappa, \tag{4.6}$$

where τ is a local stabilization parameter, see (Cockburn et al., 2008) for details.

With the definitions of numerical trace λ_h and the numerical flux $\widehat{\mathbf{q}}\cdot\mathbf{n}$, this discrete

problem resulting from HDG discretization can be written as: to find $(\mathbf{q}_h, u_h, \lambda_h) \in \mathbf{V} \times W \times \Lambda$ such that for all $(\boldsymbol{v}_h, w_h, \mu_h) \in \mathbf{V} \times W \times \Lambda$

$$\begin{cases} -(\rho \mathbf{q}_h, \mathbf{v}_h)_{\mathcal{T}_h} - (u_h, \nabla \cdot \mathbf{v}_h)_{\mathcal{T}_h} + \langle \lambda_h, \mathbf{v}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0 \\ (\mathbf{q}_h, \nabla w_h)_{\mathcal{T}_h} - \langle \widehat{\mathbf{q}}_h \cdot \boldsymbol{n}, w_h \rangle_{\partial \mathcal{T}_h} &= (f, w_h)_\Omega \\ \langle \widehat{\mathbf{q}}_h \cdot \boldsymbol{n}, \mu_h \rangle_{\partial \mathcal{T}_h} &= 0, \end{cases} \tag{4.7}$$

where $(\cdot,\cdot)_{\mathcal{T}_h} = \sum_{\kappa \in \mathcal{T}_h} (\cdot,\cdot)_\kappa$ and $\langle \cdot,\cdot \rangle_{\partial \mathcal{T}_h} = \sum_{\kappa \in \mathcal{T}_h} \langle \cdot,\cdot \rangle_{\partial \kappa}$.

Correspondingly, the matrix form of (4.7) is

$$\begin{bmatrix} A_{\mathbf{qq}} & A_{\mathbf{q}u} & A_{\mathbf{q}\lambda} \\ A_{\mathbf{q}u}^T & A_{uu} & A_{u\lambda} \\ A_{\mathbf{q}\lambda}^T & A_{u\lambda}^T & A_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \\ 0 \end{bmatrix}, \tag{4.8}$$

where we use $\mathbf{q}$, u, and λ to denote the unknowns associated with $\mathbf{q}_h$, u_h, and λ_h, respectively.

In each κ, given the value of λ on $\partial\kappa$, $\mathbf{q}_h$ and u_h can be uniquely determined; see (Cockburn et al., 2009a). Namely, given λ_h, the solution $(\mathbf{q}_h, u_h)$ of (4.5) is uniquely determined. In the matrix form, we note that

$$\begin{bmatrix} A_{\mathbf{qq}} & A_{\mathbf{q}u} \\ A_{\mathbf{q}u}^T & A_{uu} \end{bmatrix}$$

is block diagonal, each block is nonsingular and corresponding to one element κ. Therefore, we can easily eliminate $\mathbf{q}$ and u in each element independently from (4.8) and obtain the system for λ only

$$A\lambda = b, \tag{4.9}$$

where

$$A = A_{\lambda\lambda} - [A_{\mathbf{q}\lambda}^T \; A_{u\lambda}^T] \begin{bmatrix} A_{\mathbf{qq}} & A_{\mathbf{q}u} \\ A_{\mathbf{q}u}^T & A_{uu} \end{bmatrix}^{-1} \begin{bmatrix} A_{\mathbf{q}\lambda} \\ A_{u\lambda} \end{bmatrix}$$

and

$$b = -[A_{\mathbf{q}\lambda}^T \; A_{u\lambda}^T] \begin{bmatrix} A_{\mathbf{qq}} & A_{\mathbf{q}u} \\ A_{\mathbf{q}u}^T & A_{uu} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ F_h \end{bmatrix}.$$

Once the solution of (4.9) is obtained, the solution of (4.8) can be completed by computing $\mathbf{q}$ and u in each element with given λ.

By (Cockburn et al., 2009a, Theorem 2.1), the system (4.9) can be considered as the matrix form of the following problem: to find $\lambda \in \Lambda$ such that

$$a_h(\lambda, \mu) = b_h(\mu), \quad \forall\ \mu \in \Lambda. \tag{4.10}$$

Here

$$\begin{array}{rcl} a_h(\eta,\mu) & = & \sum_{K\in\mathcal{T}_h} a_K(\eta,\mu) = \sum_{K\in\mathcal{T}_h} (\rho\mathcal{Q}\eta, \mathcal{Q}\mu)_K + \langle \tau(\mathcal{U}\eta - \eta), (\mathcal{U}\mu - \mu)\rangle_{\partial K}\,, \\ b_h(\mu) & = & \sum_{K\in\mathcal{T}_h} b_K(\eta,\mu) = \sum_{K\in\mathcal{T}_h} (f_h, \mathcal{U}\mu)_K, \end{array} \tag{4.11}$$

where $\mathcal{Q}\mu$ and $\mathcal{U}\mu$ are the unique solution ($\mathcal{Q}\mu = \mathbf{q}_h$,$\mathcal{U}\mu = u_h$) of the local element problem (4.5) with $\lambda = \mu$.

In (Cockburn et al., 2014, Theorem 3.6), the bilinear form $a_h(\cdot,\cdot)$ has been proved to be positive definite. More properties of $a_h(\cdot,\cdot)$ will be studied in Subsection 4.1.5.

In next two subsections, we consider to solve the system of the numerical trace λ (4.9) by a BDDC algorithm.

4.1.3 Reduced Subdomain Interface Problem

We decompose Ω into N nonoverlapping subdomain Ω_i with diameters H_i, $i = 1,\ldots,N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse rectangles/hexahedra and the numbers of such elements in the corresponding sudomains are uniformly bounded. We note that the algorithm can be extended to different types of subdomains. Also, we assume $a(x)$, the coefficient of (4.1), is constant in each subdomain.

We define edges/faces as open sets shared by two subdomains. Two nodes belong to the same face when they are associated with the same pair of subdomains. We then introduce quasi-uniform triangulation of each subdomain. Let Γ be the interface between the subdomains. And let the set of interface nodes be denoted by $\Gamma_h := \left(\cup_i \partial\Omega_{i,h}\right) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is that of $\partial\Omega$.

We can decompose Λ into the subdomain interior and interface parts as below:

$$\Lambda = \Lambda_I \oplus \hat{\Lambda}_\Gamma,$$

where Λ_I is the product space of subdomain interior degrees of freedom, i.e., $\Lambda_I = \Pi_{i=1}^N \Lambda_I^{(i)}$, and $\hat{\Lambda}_\Gamma$ denotes the set of interface degrees of freedom associated with finite element solutions which are continuous across the subdomain interface.

The global trace problem is to find $(\lambda_I, \lambda_\Gamma) \in (\Lambda_I, \hat{\Lambda}_\Gamma)$, such that

$$\begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{I\Gamma}^T & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \lambda_I \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} b_I \\ b_\Gamma \end{bmatrix} \tag{4.12}$$

We denote the subdomain interface numerical trace space $\Lambda_\Gamma^{(i)}$, and the associated product space by $\Lambda_\Gamma = \Pi_{i=1}^N \Lambda_\Gamma^{(i)}$. We define the restriction operators $R_\Gamma^{(i)} : \hat{\Lambda}_\Gamma \to \Lambda_\Gamma^{(i)}$ to be an operator which maps functions in the continuous global interface numerical trace space $\hat{\Lambda}_\Gamma$ to the subdomain component space $\Lambda_\Gamma^{(i)}$. Also, $R_\Gamma : \hat{\Lambda}_\Gamma \to \Lambda_\Gamma$ is the direct sum of $R_\Gamma^{(i)}$.

The global problem (4.12) is assembled from subdomain problems

$$A^{(i)} \lambda^{(i)} = b^{(i)},$$

where

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{I\Gamma}^{(i)T} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}, \lambda^{(i)} = \begin{bmatrix} \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} \in \begin{bmatrix} \Lambda_I^{(i)} \\ \Lambda_\Gamma^{(i)} \end{bmatrix}, \text{and } b^{(i)} = \begin{bmatrix} b_I^{(i)} \\ b_\Gamma^{(i)} \end{bmatrix}.$$

We can eliminate the subdomain interior variables $\lambda_I^{(i)}$ in each subdomain independently and define the subdomain Schur complement $S_\Gamma^{(i)}$ by: given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, determine $S_\Gamma^{(i)} u_\Gamma^{(i)}$ such that

$$\begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{I\Gamma}^{(i)T} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ S_\Gamma^{(i)} \lambda_\Gamma^{(i)} \end{bmatrix}. \tag{4.13}$$

We denote the direct sum of the $S_\Gamma^{(i)}$ by S_Γ, i.e.,

$$S_\Gamma = \begin{bmatrix} S_\Gamma^{(1)} & & \\ & \ddots & \\ & & S_\Gamma^{(N)} \end{bmatrix}.$$

The global interface problem is assembled from the subdomain interface problems, and can be written as: find $\lambda_\Gamma \in \hat{\Lambda}_\Gamma$,such that

$$\widehat{S}_\Gamma \lambda_\Gamma = b_\Gamma, \tag{4.14}$$

where $b_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)T} b_\Gamma^{(i)}$,and $\hat{S}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)T} S_\Gamma^{(i)} R_\Gamma^{(i)}$. Here, $\hat{S}_\Gamma$ is a symmetric, positive definite operator defined on the interface space $\widehat{\Lambda}_\Gamma$. We will propose a BDDC preconditioner for solving (4.14) with a preconditioned conjugate gradient method.

4.1.4 The BDDC Preconditioner

We introduce a partially assembled interface space $\widetilde{\Lambda}_\Gamma$ by

$$\tilde{\Lambda}_\Gamma = \hat{\Lambda}_\Pi \oplus \Lambda_\Delta = \hat{\Lambda}_\Pi \oplus \prod_{i=1}^N \Lambda_\Delta^{(i)}.$$

Here, $\hat{\Lambda}_\Pi$ is the coarse level, primal interface space which is spanned by subdomain interface edge/face basis functions with constant values at the nodes of the edge/face for two/three dimensions. We change the variables so that the degree of freedom(dof) of each primal

constraint is explicit; see (Li & Widlund, 2006b; Klawonn & Widlund, 2006). The space $\Lambda_\triangle$ is the direct sum of the $\Lambda_\triangle^{(i)}$, which are spanned by the dual interface dofs with zero average over each edge/face. In the space $\tilde{\Lambda}_\Gamma$, we relax continuity constraints on the dual variables but retain all primal continuity constraints, which makes all the linear systems nonsigular.

Before discussing the BDDC preconditioner, we first introduce several restriction, extension, and scaling operators between different spaces. Specifically, $\overline{R}_\Gamma^{(i)} : \tilde{\Lambda}_\Gamma \to \Lambda_\Gamma^{(i)}$, which restricts functions in the space $\widetilde{\Lambda}_\Gamma$ to the components $\Lambda_\Gamma^{(i)}$. $\overline{R}_\Gamma : \widetilde{\Lambda}_\Gamma \to \Lambda_\Gamma$, which is the direct sum of $\overline{R}_\Gamma^{(i)}$. $R_\Delta^{(i)} : \hat{\Lambda}_\Gamma \to \Lambda_\Delta^{(i)}$, which maps functions from $\widehat{\Lambda}_\Gamma$ to $\Lambda_\Delta^{(i)}$. $R_{\Gamma\Pi} : \widehat{\Lambda}_\Gamma \to \widehat{\Lambda}_\Pi$, which is a restriction operator from $\widehat{\Lambda}_\Gamma$ to its subspace $\widehat{\Lambda}_\Pi$. $R_\Pi^{(i)} : \widehat{\Lambda}_\Pi \to \Lambda_\Pi^{(i)}$, which maps vectors in $\widehat{\Lambda}_\Pi$ into their components in $\Lambda_\Pi^{(i)}$. $\widetilde{R}_\Gamma : \widehat{\Lambda}_\Gamma \to \widetilde{\Lambda}_\Gamma$, which is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define the positive scaling factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in \left[\frac{1}{2}, \infty\right)$,

$$\delta_i^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j\in\mathcal{N}_x} \rho_j^\gamma(x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\rho_i(x) = 1/a(x)$, $a(x)$ is the entry of $\mathbf{a}(x)$ in the ith subdomain, and $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. Since $\rho_i(x)$ is constant in each subdomain, thus $\delta_i^\dagger(x)$ is constant on each edge/face. We are now ready to define scaled operators. $R_{D,\Delta}^{(i)}$ can be obtained by multiplying each row of $R_\Delta^{(i)}$ with the scaling operator $\delta_i^\dagger(x)$. The scaled operator $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$. Furthermore, $\widetilde{R}_\Delta^{(i)} : \widetilde{\Lambda}_\Gamma \to \Lambda_\Delta^{(i)}$, $\widetilde{R}_{\Gamma\Pi} : \widetilde{\Lambda}_\Gamma \to \hat{\Lambda}_\Pi$.

We define the partial assembled interface Schur complement $\widetilde{S}_\Gamma$ by $\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma$. Note that we can obtain the fully assembled Schur complement $\widehat{S}_\Gamma$ by a further assembly, i.e., $\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma$. Therefore, the reduced interface problem can be written as: find $\lambda_\Gamma \in \widehat{W}_\Gamma$ such that

$$\widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \lambda_\Gamma = b_\Gamma.$$

The preconditioned BDDC method is of the form

$$M_{BDDC}^{-1}\hat{S}_\Gamma\lambda_\Gamma = M_{BDDC}^{-1}b_\Gamma, \tag{4.15}$$

where the preconditioner is defined as:

$$M_{BDDC}^{-1} = \widetilde{R}_{D,\Gamma}^T\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}. \tag{4.16}$$

Note that the inverse of the partial assembled Schur complement operator $\widetilde{S}_\Gamma^{-1}$ on the space $\tilde{\Lambda}_\Gamma$ can be calculated explicitely by linear algebra as below:

$$\widetilde{S}_\Gamma^{-1} = R_{\Gamma\Delta}^T\left(\sum_{i=1}^{N}\begin{bmatrix} 0 & R_\Delta^{(i)T}\end{bmatrix}\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\triangle I}^{(i)} & A_{\Delta\Delta}^{(i)}\end{bmatrix}^{-1}\begin{bmatrix} 0 \\ R_\Delta^{(i)}\end{bmatrix}\right)R_{\Gamma\Delta} + \Phi S_\Pi^{-1}\Phi^T,$$

where

$$\Phi = R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T\left(\sum_{i=1}^{N}\begin{bmatrix} 0 & R_\Delta^{(i)T}\end{bmatrix}\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\triangle I}^{(i)} & A_{\Delta\Delta}^{(i)}\end{bmatrix}^{-1}\begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T}\end{bmatrix}R_\Pi^{(i)}\right),$$

and

$$S_\Pi = \sum_{i=1}^{N}R_\Pi^{(i)T}\left(A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)}\end{bmatrix}\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\triangle I}^{(i)} & A_{\Delta\Delta}^{(i)}\end{bmatrix}^{-1}\begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T}\end{bmatrix}\right)R_\Pi^{(i)}.$$

Here, subscripts I, $\triangle$, and Π indicates the interior, dual, and primal variables, respectively. For details, refer to (Tu, 2006).

4.1.5 Some Auxiliary Results

In this section, we collect a number of results that are needed in our condition number estimate of the preconditioned system (4.15). We define

$$\gamma_{h,\tau} = \max_{K\in\mathcal{T}_h} \{1+\tau_K h_K\}, \tag{4.17}$$

where τ_K and h_K are the stabilized parameter and diameter of the element K, respectively. We use c and C to denote constants that are independent of h, H, τ_K and the coefficient ρ of (4.4).

We first introduce several useful norms, which are defined in (Cockburn et al., 2014; Gopalakrishnan, 2003). For any domain D, we denote the $L^2(D)$ norm by $\|\cdot\|_D$. For any numerical trace $\lambda \in \Lambda(D)$, define

$$|||\lambda|||_D^* = \left(\frac{1}{h} \sum_{K\in\mathcal{T}_h, K\subseteq\bar{D}} \|\lambda - m_K(\lambda)\|^2_{L^2(\partial K)} \right)^{\frac{1}{2}} \tag{4.18}$$

where m_k is the average of the trace defined by $m_K(\lambda) = \frac{1}{|\partial K|}\int_{\partial K} \lambda ds$, and $|\partial K|$ is the measure (the length for 2D and area for 3D) of the boundary of K. We note that when D is strictly contained Ω, $|||\lambda|||_D^*$ is a semi-norm. When $D=\Omega$, we use the simple notation $|||\lambda|||^*$ for $|||\lambda|||_\Omega^*$. $|||\lambda|||^*$ is an H^1-like norm, since the functions in Λ having zero boundary conditions on $\partial\Omega$.

We recall the bilinear form $a_h(\eta,\mu)$ in (4.11) and define the norm

$$|\lambda|_A^2 = a_h(\lambda,\lambda), \forall \lambda \in \Lambda.$$

Given a subdomain Ω_i, let $a_h^{(i)}(\cdot,\cdot)$ be the restriction of $a_h(\cdot,\cdot)$ to Ω_i, and we can define similar norms. Let

$$\left|\lambda^{(i)}\right|^2_{A^{(i)}} = a_h^{(i)}\left(\lambda^{(i)},\lambda^{(i)}\right), \qquad \forall \lambda^{(i)} \in \Lambda^{(i)}.$$

The global norm $|\lambda|_A$ can be assembled from the subdomain norms as

$$|\lambda|_A^2 = \sum_{i=1}^{N} \left|\lambda^{(i)}\right|_{A^{(i)}}^2$$

where $\lambda^{(i)} = R_\Gamma^{(i)}\lambda$, i.e., the restriction of λ to the subdomain Ω_i. The following lemma is in (Cockburn et al., 2014, Theorem 3.9) applied to each subdomain Ω_i.

Lemma 4.1.1. *For any* $\lambda^{(i)} \in \Lambda^{(i)}$,

$$c\rho_i |||\lambda|||_h^{*,2} \leq |\lambda|_A^2 \leq C\rho_i\gamma_{h,\tau}|||\lambda|||_h^{*,2},$$

where $\gamma_{h,\tau}$ *is defined in* (4.17).

Given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, we can define a harmonic extension $\mathcal{H}^{(i)}\left(\lambda_\Gamma^{(i)}\right) : \Lambda^{(i)} \to \Lambda^{(i)}$ as

$$|\ \mathcal{H}^{(i)}\left(\lambda_\Gamma^{(i)}\right)\ |_{A^{(i)}}^2 = \min_{\lambda^{(i)}\in\Lambda^{(i)},\, \lambda^{(i)}=\lambda_\Gamma^{(i)} \text{on}\, \partial\Omega_i} |\ \lambda^{(i)}\ |_{A^{(i)}}^2 . \tag{4.19}$$

By the definition of $\mathcal{H}^{(i)}$ and (4.13), we have

$$|\ \lambda_\Gamma^{(i)}\ |_{S_\Gamma^{(i)}}^2 := \left(\lambda_\Gamma^{(i)}\right)^T S_\Gamma^{(i)}\lambda_\Gamma^{(i)} = |\ \mathcal{H}^{(i)}(\lambda_\Gamma^{(i)})\ |_{A^{(i)}}^2 .$$

The bilinear form $a_h(\cdot,\cdot)$ defined in (4.11) is closely related to the bilinear form of the Lagrange multiplier of the hybridized mixed finite element, (Cockburn et al., 2014; Gopalakrishnan, 2003). Here we denote the corresponding bilinear form and norms with a superscript RT, referring to the Raviart-Thomas finite element of the same order of the HDG method. We list some results which are useful in our analysis. The following lemma is in (Gopalakrishnan, 2003, Theorem 2.2) applied to each subdomain Ω_i:

Lemma 4.1.2. *For any* $\lambda \in \Lambda$,

$$c\rho_i |||\lambda|||_{\Omega_i}^{*,2} \leq |\ \lambda\ |_{A^{RT}}^2 \leq C\rho_i |||\lambda|||_{\Omega_i}^{*,2}.$$

Given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, we can similarly define a harmonic extension $\mathcal{H}^{RT^{(i)}}\left(\lambda_\Gamma^{(i)}\right) : \Lambda^{(i)} \to \Lambda^{(i)}$ as

$$| \mathcal{H}^{RT^{(i)}}\left(\lambda_\Gamma^{(i)}\right) |_{A^{RT^{(i)}}} = \min_{\lambda^{(i)} \in \Lambda^{(i)},\, \lambda^{(i)} = \lambda_\Gamma^{(i)} \text{on } \partial\Omega_i} | \lambda^{(i)} |_{A^{RT^{(i)}}}$$

and have

$$| \lambda_\Gamma^{(i)} |_{S_\Gamma^{RT^{(i)}}} := \sqrt{\left(\lambda_\Gamma^{(i)}\right)^T S_\Gamma^{RT^{(i)}} \lambda_\Gamma^{(i)}} = | \mathcal{H}^{RT^{(i)}}\left(\lambda_\Gamma^{(i)}\right) |_{A^{RT^{(i)}}} .$$

Let $\Lambda^{0,(i)}$ be the zero-order numerical trace space in Ω_i and Q_0 be the L^2-orthogonal projection from $\Lambda^{(i)}$ into $\Lambda^{0,(i)}$. By a scaling argument, see (Gopalakrishnan, 2003, (4.9) and (4.10)), we have the following lemma:

Lemma 4.1.3. *For any $\lambda^{(i)} \in \Lambda^{(i)}$,*

$$\left\|\left\| Q_0 \lambda^{(i)} \right\|\right\|_{\Omega_i}^{*} \leq C \left\|\left\| \lambda^{(i)} \right\|\right\|_{\Omega_i}^{*} \tag{4.20}$$

and

$$\sum_{K \in \mathcal{T}_h, K \subseteq \Omega_{(i)}} \left\| \lambda^{(i)} - Q_0 \lambda^{(i)} \right\|_{L^2(\partial K)}^2 \leq Ch \left\|\left\| \lambda \right\|\right\|_{\Omega_{(i)}}^{*,2}. \tag{4.21}$$

Given a subdomain Ω_i, we define partition of unity functions associated with its edges/-faces. An edge/face in the interface Γ only belongs to exactly two subdomains. We denote the face shared by Ω_i and Ω_j by F^{ij}. Let $\zeta_{F^{ij}}$ be the characteristic function of F^{ij}, i.e., the function that is identically one on F_h^{ij} and zero on $\partial\Omega_i^h \setminus F_h^{ij}$, where F_h^{ij} contains the degrees of freedom of Ω_i^h on $F^{ij} \subset \partial\Omega_i$. We clearly have

$$\sum_{F^{ij} \subset \partial\Omega_i} \zeta_{F^{ij}}(x) = 1, \qquad \lambda_\Gamma^{(i)} = \sum_{F^{ij} \subset \partial\Omega_i} \zeta_{F^{ij}}(x)\, \lambda_\Gamma^{(i)},$$

for any $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, the numerical trace space on $\partial\Omega_i$.

Let $\bar{\lambda}_{F^{ij}}^{(i)} = \frac{1}{|F^{ij}|} \int_{F^{ij}} \lambda_\Gamma^{(i)} dx$, the average of $\lambda_\Gamma^{(i)}$ over F^{ij}. Particularly, we have the following

lemma for the Lagrange multiplier of the zero-order hybridized mixed finite element, which can be proved using (Tu, 2005, Lemmas 5.4 and 5.5).

Lemma 4.1.4. *For any* $\lambda_\Gamma^{0,(j)} \in \Lambda_\Gamma^{0,(j)}$ *we have*

$$\delta_j^{\dagger 2} \left| \zeta_{\mathcal{F}^{ij}} \left(\lambda_\Gamma^{0,(j)} - \bar{\lambda}_\Gamma^{0,(j)} \right) \right|^2_{S_\Gamma^{RT(i)}} \leq C \left(1 + log \frac{H}{h} \right)^2 \left| \lambda_\Gamma^{0,(j)} \right|_{S_\Gamma^{RT(j)}}$$

We define the interface averaging operator E_D by

$$E_D = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T,$$

which computes a weighted average across the subdomain interface Γ and then distributes the averages to the degrees of freedom on the boundary of the subdomain. The interface averaging operator E_D satisfies the following bound:

Lemma 4.1.5. *For any* $\lambda_\Gamma \in \widetilde{\Lambda}_\Gamma$, $\mid E_D \lambda_\Gamma \mid^2_{\widetilde{S}_\Gamma} \leq C \gamma_{h,\tau} \left(1 + log \frac{H}{h} \right)^2 \mid \lambda_\Gamma \mid^2_{\widetilde{S}_\Gamma}$ *where* $\gamma_{h,\tau}$ *is defined in (4.17).*

Proof. Given any $\lambda_\Gamma \in \widetilde{\Lambda}_\Gamma$, we have

$$\begin{aligned} \mid E_D \lambda_\Gamma \mid^2_{\widetilde{S}_\Gamma} \leq & \mid \lambda_\Gamma \mid^2_{\widetilde{S}_\Gamma} + \mid \lambda_\Gamma - E_D \lambda_\Gamma \mid^2_{\widetilde{S}_\Gamma} \\ = & \mid \lambda_\Gamma \mid^2_{\widetilde{S}_\Gamma} + \mid \overline{R}_\Gamma \left(\lambda_\Gamma - E_D \lambda_\Gamma \right) \mid^2_{S_\Gamma} \\ = & \mid \lambda_\Gamma \mid^2_{\widetilde{S}_\Gamma} + \sum_{i=1}^{N} \mid \overline{R}_\Gamma^{(i)} \left(\lambda_\Gamma - E_D \lambda_\Gamma \right) \mid^2_{S_\Gamma^{(i)}} . \end{aligned}$$

Let $v_i := \overline{R}_\Gamma^{(i)} \left(\lambda_\Gamma - E_D \lambda_\Gamma \right)$, which indicates the restriction of $\lambda_\Gamma - E_D \lambda_\Gamma$ to the subdomain Ω_i. It follows that

$$\begin{aligned} \zeta_{F^{ij}} v_i = \left(\lambda_\Gamma - E_D \lambda_\Gamma \right) \mid_{F^{ij}} = \zeta_{F^{ij}} \lambda_\Gamma^{(i)} - \zeta_{F^{ij}} E_D \lambda_\Gamma = \zeta_{F^{ij}} \left(\lambda_\Gamma^{(i)} - \delta_i^\dagger \lambda_\Gamma^{(i)} - \delta_j^\dagger \lambda_\Gamma^{(j)} \right) \\ = \zeta_{F^{ij}} \delta_j^\dagger \left(\lambda_\Gamma^{(i)} - \lambda_\Gamma^{(j)} \right), \end{aligned}$$

where $\lambda_\Gamma^{(i)} = R_\Gamma^{(i)}\lambda_\Gamma$ and $\lambda_\Gamma^{(j)} = R_\Gamma^{(j)}\lambda_\Gamma$.

Also, $|\ v_i\ |^2_{S_\Gamma^{(i)}} = \left|\sum_{F^{ij}\subset\partial\Omega_i}\zeta_{F^{ij}}v_i\right|^2_{S_\Gamma^{(i)}} \leq \sum_{F^{ij}\subset\partial\Omega_i}|\zeta_{F^{ij}}v_i|^2_{S_\Gamma^{(i)}}$.

We want to show that $|\ \zeta_{F^{ij}}v_i\ |^2_{S_\Gamma^{(i)}} \leq C\gamma_{h,\tau}\left(1+log\frac{H}{h}\right)^2\left(|\ \lambda_\Gamma^{(i)}\ |_{S_\Gamma^{(i)}} + |\ \lambda_\Gamma^{(j)}\ |_{S_\Gamma^{(j)}}\right)$.

Let $\lambda^{(i)} = \mathcal{H}^{(i)}\left(\lambda_\Gamma^{(i)}\right)$ and $\lambda^{(j)} = \mathcal{H}^{(j)}\left(\lambda_\Gamma^{(j)}\right)$. Then it follows that

$$\left|\lambda_\Gamma^{(i)}\right|_{S_\Gamma^{(i)}} = \left|\lambda^{(i)}\right|_{A^{(i)}}, \quad \text{and} \quad \left|\lambda_\Gamma^{(j)}\right|_{S_\Gamma^{(j)}} = \left|\lambda^{(j)}\right|_{A^{(j)}}.$$

We note that the simple inequality

$$\rho_i\delta_j^{\dagger 2} \leq \min(\rho_i, \rho_j) \tag{4.22}$$

holds for $\gamma \in [1/2, \infty)$ (Klawonn & Widlund, 2006). Let $\bar{\lambda}_{F^{ij}}^{(i)} = \frac{1}{|F^{ij}|}\int_{F^{ij}}\lambda_\Gamma^{(i)}dx$. It is easy to see that $\bar{\lambda}_{F^{ij}}^{(i)} = \bar{\lambda}_{F^{ij}}^{(j)}$.

Note that:

$$\begin{aligned}
|\zeta_{F^{ij}}v_i|^2_{S_\Gamma^{(i)}} &= |\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}v_i\right)|^2_{A^{(i)}} = |\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}\delta_j^\dagger\left(\lambda_\Gamma^{(i)} - \lambda_\Gamma^{(j)}\right)\right)|^2_{A^{(i)}} \\
&\leq \delta_j^{\dagger 2}\left(|\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}\left(\lambda_\Gamma^{(i)} - \bar{\lambda}_{F^{ij}}^{(i)}\right)\right)|^2_{A^{(i)}} + |\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}\left(\lambda_\Gamma^{(j)} - \bar{\lambda}_{F^{ij}}^{(j)}\right)\right)|^2_{A^{(i)}}\right).
\end{aligned}$$

We only need to estimate the second term above, and the first term can be estimated similarly. Let $\lambda^{0,(j)} = Q_0\lambda^{(j)} \in \Lambda^{0,(j)}$ and $\lambda_\Gamma^{0,(j)}$ is the restriction of $\lambda^{0,(j)}$ to $\partial\Omega_j$. We have

$$\begin{aligned}
\delta_j^{\dagger 2}|\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}\left(\lambda_\Gamma^{(j)} - \bar{\lambda}_{F^{ij}}^{(j)}\right)\right)|^2_{A^{(i)}} &= \delta_j^{\dagger 2}|\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}\left(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)} + \lambda_\Gamma^{0,(j)} - \bar{\lambda}_{F^{ij}}^{(j)}\right)\right)|^2_{A^{(i)}} \\
&\leq \delta_j^{\dagger 2}|\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}\left(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)}\right)\right)|^2_{A^{(i)}} + \delta_j^{\dagger 2}|\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}\left(\lambda_\Gamma^{0,(j)} - \bar{\lambda}_{F^{ij}}^{(j)}\right)\right)|^2_{A^{(i)}}.
\end{aligned} \tag{4.23}$$

We estimate the above two terms in (4.23) separately. Let $\mathcal{R}^{(i)}\left(\lambda_\Gamma^{(i)}\right) : \Lambda_\Gamma^{(i)} \to \Lambda^{(i)}$ be

the zero extension of $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$ to $\Lambda^{(i)}$.

$$
\begin{aligned}
&\delta_j^{\dagger 2} |\mathcal{H}^{(i)} \left(\zeta_{F^{ij}} \left(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)} \right) \right)|_{A^{(i)}}^2 \leq C \delta_j^{\dagger 2} |\mathcal{R}^{(i)} \left(\zeta_{F^{ij}} \left(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)} \right) \right)|_{A^{(i)}}^2 \\
&\leq C \gamma_{h,\tau} \delta_j^{\dagger 2} \rho_i \left| \left| \left| \mathcal{R}^{(i)} \left(\zeta_{F^{ij}} \left(\lambda_\Gamma^{(j)} - \lambda_\Gamma^{0,(j)} \right) \right) \right| \right|_{\Omega_i}^{*,2} \\
&\leq C \gamma_{h,\tau} \min\left(\rho_i, \rho_j\right) h^{-1} \sum_{K \in \mathcal{T}_h, K \subseteq \Omega_i} \| \zeta_{F^{ij}} \left(\lambda^{(j)} - \lambda^{0,(j)} \right) \|_{L_2(\partial K)}^2 \\
&= C \gamma_{h,\tau} \min\left(\rho_i, \rho_j\right) h^{-1} \sum_{e \subseteq (\partial K \cap F^{ij}), K \in \mathcal{T}_h, K \subseteq \Omega_i} \| \lambda^{(j)} - \lambda^{0,(j)} \|_{L_2(e)}^2 \\
&= C \gamma_{h,\tau} \min\left(\rho_i, \rho_j\right) h^{-1} \sum_{e \subseteq (\partial K \cap F^{ij}), K \in \mathcal{T}_h, K \subseteq \Omega_j} \| \lambda^{(j)} - \lambda^{0,(j)} \|_{L_2(e)}^2
\end{aligned}
$$

Here, we use the definition of $\mathcal{H}^{(i)}$ and $\mathcal{R}^{(i)}$ for the first inequality. Lemma 4.1.1 is used for for the second inequality. (4.22) and the definition of $|||\cdot|||^*$ in (4.18) are used for the third inequality. Further, we have

$$
\begin{aligned}
&C \gamma_{h,\tau} \min\left(\rho_i, \rho_j\right) h^{-1} \sum_{e \subseteq (\partial K \cap F^{ij}), K \in \mathcal{T}_h, K \subseteq \Omega_j} \| \lambda^{(j)} - \lambda^{0,(j)} \|_{L_2(e)}^2 \\
&\leq C \gamma_{h,\tau} \rho_j h^{-1} \left(\sum_{K \in \mathcal{T}_h, K \subseteq \Omega_j} \| \lambda^{(j)} - \lambda^{0,(j)} \|_{L_2(\partial K)}^2 \right) \\
&\leq C \gamma_{h,\tau} \rho_j h^{-1} h \left| \left| \left| \lambda^{(j)} \right| \right| \right|_{\Omega_j}^{*,2} \leq C \gamma_{h,\tau} |\lambda^{(j)}|_{A^{(j)}}^2 = C \gamma_{h,\tau} |\lambda_\Gamma^{(j)}|_{S_\Gamma^{(j)}}^2 .
\end{aligned}
$$

We used (4.21) in Lemma 4.1.3 for the last inequality. For the second term in (4.23), we have

$$
\begin{aligned}
&\delta_j^{\dagger 2} |\mathcal{H}^{(i)} \left(\zeta_{F^{ij}} \left(\lambda_\Gamma^{0,(j)} - \bar{\lambda}_{F^{ij}}^{(j)} \right) \right)|_{A^{(i)}}^2 \leq C \delta_j^{\dagger 2} |\mathcal{H}_{RT}^{(i)} \left(\zeta_{F^{ij}} \left(\lambda_\Gamma^{0,(j)} - \bar{\lambda}_{F^{ij}}^{(j)} \right) \right)|_{A^{(i)}}^2 \\
&\leq C \gamma_{h,\tau} \delta_j^{\dagger 2} |\mathcal{H}_{RT}^{(i)} \left(\zeta_{F^{ij}} \left(\lambda_\Gamma^{0,(j)} - \bar{\lambda}_{F^{ij}}^{(j)} \right) \right)|_{A_{RT}^{(i)}}^2 \\
&= C \gamma_{h,\tau} \delta_j^{\dagger 2} |\zeta_{F^{ij}} \left(\lambda_\Gamma^{0,(j)} - \bar{\lambda}_{F^{ij}}^{(j)} \right)|_{S_{\Gamma,RT}^{(i)}}^2
\end{aligned}
$$

Here the definition of $\mathcal{H}^{(i)}$ and $\mathcal{H}^{RT(i)}$ are used for the first inequality. Lemmas 4.1.1 and

4.1.2 are used for the second inequality and definition of $\mathcal{H}^{RT^{(i)}}$ is used for the last equality. By the equivalence lemmas Lemma 4.1.1 and 4.1.2 for the zeroth-order Lagrange multipliers for the hybridized mixed finite element method and Lemma 4.1.4, and from the observation that $\bar{\lambda}^{(j)}_{F^{ij}} = \bar{\lambda}^{0,(j)}_{F^{ij}}$, we have

$$
\begin{aligned}
&\delta_j^{\dagger 2}|\mathcal{H}^{(i)}\left(\zeta_{F^{ij}}\left(\lambda_\Gamma^{0,(j)} - \bar{\lambda}^{(j)}_{F^{ij}}\right)\right)|^2_{A^{(i)}} \leq C\gamma_{h,\tau}\delta_j^{\dagger 2}|\zeta_{F^{ij}}\left(\lambda_\Gamma^{0,(j)} - \bar{\lambda}^{0,(j)}_{F^{ij}}\right)|^2_{S^{(i)}_{\Gamma,\,RT}}\\
&\leq C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\lambda_\Gamma^{0,(j)}|^2_{S^{(j)}_{\Gamma,\,RT}} = C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\mathcal{H}^{(j)}_{RT}\left(\lambda_\Gamma^{0,(j)}\right)|^2_{A^{(j)}_{RT}}\\
&\leq C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\lambda^{0,(j)}|^2_{A^{(j)}_{RT}}\\
&\leq C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2\rho_j\left|\!\left|\!\left|\lambda^{0,(j)}\right|\!\right|\!\right|^{*,2}_{\Omega_j} = C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2\rho_j\left|\!\left|\!\left|Q_0\lambda^{(j)}\right|\!\right|\!\right|^{*,2}_{\Omega_j}\\
&\leq C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2\rho_j\left|\!\left|\!\left|\lambda^{(j)}\right|\!\right|\!\right|^{*,2}_{\Omega_j}\\
&\leq C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\lambda^{(j)}|^2_{A^{(j)}} = C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\mathcal{H}^{(j)}(\lambda_\Gamma^{(j)})|^2_{A^{(j)}}\\
&= C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\lambda_\Gamma^{(j)}|^2_{S^{(j)}_\Gamma}.
\end{aligned}
$$

Here we use Lemma 4.1.4 for the second inequality. The definition $\mathcal{H}^{RT^{(i)}}$ is used for the third inequality. Lemma 4.1.2 is used for the fourth inequality. Equation (4.20) in Lemma 4.1.3 is used for the fifth inequality and Lemma 4.1.1 is used for the sixth inequality.

□

4.1.6 Conditioner Number Estimate for the BDDC Preconditioner

We are now ready to formulate and prove our main results. It follows by proving the lower and upper bound for $\lambda_\Gamma^T\hat{S}_\Gamma\lambda_\Gamma$ using Lemma 4.1.5. See similar proof as in (Li & Widlund, 2006a; Mandel et al., 2005; Tu, 2006, 2007d,c).

Theorem 4.1.6. *The condition number of the preconditioned operator $M^{-1}\hat{S}_\Gamma$ is bounded by $C\left(1+\log\left(\frac{H}{h}\right)\right)^2$, where $\gamma_{h,\tau}$ is defined in (4.17).*

Proof. It is sufficient to prove that for any $\lambda_\Gamma \in \hat{\Lambda}_\Gamma$,

$\lambda_\Gamma^T M \lambda_\Gamma \le \lambda_\Gamma^T \hat{S}_\Gamma \lambda_\Gamma \le C\left(1+\log\left(\frac{H}{h}\right)\right)^2 \lambda_\Gamma^T M \lambda_\Gamma$.

In what follows, we prove the lower and upper bound for $\lambda_\Gamma^T \hat{S}_\Gamma \lambda_\Gamma$ respectively.

Let $w_\Gamma \coloneqq M\lambda_\Gamma = \left(\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}\right)^{-1} \lambda_\Gamma$. Obviously, $w_\Gamma \in \hat{\Lambda}_\Gamma$.

Note that $\widetilde{R}_\Gamma^T \widetilde{R}_{D,\Gamma} = \widetilde{R}_{D,\Gamma}^T \widetilde{R}_\Gamma = I$.

The details for the proof the lower bound go as follows:

$$\begin{aligned}
\lambda_\Gamma^T M \lambda_\Gamma &= \lambda_\Gamma^T w_\Gamma = \lambda_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \\
&= \left\langle \widetilde{R}_\Gamma \lambda_\Gamma, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma} \\
&\le \left\langle \widetilde{R}_\Gamma \lambda_\Gamma, \widetilde{R}_\Gamma \lambda_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{1/2} \left\langle \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{1/2} \\
&= \left(\lambda_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \lambda_\Gamma\right)^{1/2} \left(\lambda_\Gamma^T M \lambda_\Gamma\right)^{1/2}.
\end{aligned}$$

Thus, we obtain $\lambda_\Gamma^T M \lambda_\Gamma \le \lambda_\Gamma^T \hat{S}_\Gamma \lambda_\Gamma$ by cancelling a common factor and squaring on both sides. Next, we prove the upper bound.

$$\begin{aligned}
\lambda_\Gamma^T \hat{S}_\Gamma \lambda_\Gamma &= \lambda_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \\
&= \left\langle \widetilde{R}_\Gamma \lambda_\Gamma, E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma} \\
&\le \left\langle \widetilde{R}_\Gamma \lambda_\Gamma, \widetilde{R}_\Gamma \lambda_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{1/2} \left\langle E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, E_D \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{1/2} \\
&\le C \left\langle \widetilde{R}_\Gamma \lambda_\Gamma, \widetilde{R}_\Gamma \lambda_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{1/2} \left(1+\log\left(\frac{H}{h}\right)\right) \left\langle \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma, \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma \right\rangle_{\widetilde{S}_\Gamma}^{1/2} \\
&= C\left(1+\log\left(\frac{H}{h}\right)\right) \left(\lambda_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \lambda_\Gamma\right)^{1/2} \left(w_\Gamma^T \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} w_\Gamma\right)^{1/2} \\
&= C\left(1+\log\left(\frac{H}{h}\right)\right) \left(\lambda_\Gamma^T \widehat{S}_\Gamma \lambda_\Gamma\right)^{1/2} \left(\lambda_\Gamma^T M \lambda_\Gamma\right)^{1/2}.
\end{aligned}$$

Thus, the upper bound is $\lambda_\Gamma^T \widehat{S}_\Gamma \lambda_\Gamma = C\left(1+\log\left(\frac{H}{h}\right)\right)^2 \lambda M \lambda_\Gamma$.

□

Number of Subdomains	Iterations	Condition number
4×4	5	2.22
8×8	9	2.39
12×12	8	2.33
16×16	8	2.34
20×20	8	2.33

Table 4.1: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\rho \equiv 1$, $\tau \equiv 1$, and $k = 0$.

Number of Subdomains	Iterations	Condition number
4×4	6	3.46
8×8	10	3.74
12×12	10	3.71
16×16	10	3.70
20×20	10	3.69

Table 4.2: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\rho \equiv 1$, $\tau \equiv 1$, and $k = 1$.

4.1.7 Numerical Experiments

We have applied our BDDC algorithms to the model problem (4.4) with $f = 2\pi^2 sin(\pi x) sin(\pi y)$ and $\Omega = [0, 1]^2$. We decompose the unit square into $\sqrt{N} \times \sqrt{N}$ subdomains with sidelength $H = 1/\sqrt{N}$. Equation 4.1 is discretized in each subdomain by the kth-order HDG method with element diameter h. The preconditioned conjugate gradient iteration is stopped when the l_2-norm of the residual reduced by a factor of 10^6.

In the first set of experiments, we take the coefficient $\rho \equiv 1$, and fix the size of the subdomain problem to be $\frac{H}{h} = 8$. The first six tables show the iteration counts and the estimates

Number of Subdomains	Iterations	Condition number
4×4	6	4.47
8×8	12	4.84
12×12	12	4.79
16×16	12	4.78
20×20	12	4.78

Table 4.3: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\rho \equiv 1$, $\tau \equiv 1$, and $k = 2$.

Number of Subdomains	Iterations	Condition number
4×4	5	2.35
8×8	9	2.53
12×12	9	2.50
16×16	9	2.50
20×20	8	2.46

Table 4.4: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\rho \equiv 1$, $\tau \equiv \frac{1}{h}$, and $k = 0$.

Number of Subdomains	Iterations	Condition number
4×4	6	3.30
8×8	10	3.57
12×12	10	3.53
16×16	10	3.53
20×20	10	3.52

Table 4.5: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\rho \equiv 1$, $\tau \equiv \frac{1}{h}$, and $k = 1$.

Number of Subdomains	Iterations	Condition number
4×4	6	4.38
8×8	12	4.75
12×12	12	4.68
16×16	12	4.68
20×20	12	4.68

Table 4.6: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\rho \equiv 1$, $\tau \equiv \frac{1}{h}$, and $k = 2$.

Number of Subdomains	Iterations	Condition number
4×4	3	1.80
8×8	8	2.08
12×12	9	2.13
16×16	10	2.15
20×20	10	2.16

Table 4.7: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\tau \equiv 1$, $k = 0$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

Number of Subdomains	Iterations	Condition number
4×4	4	2.35
8×8	8	2.74
12×12	9	2.82
16×16	10	2.85
20×20	10	2.86

Table 4.8: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\tau \equiv 1$, $k = 1$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

of the condition numbers for the BDDC preconditioned operator with changing subdomain numbers for different polynommial orders and stabilization parameter. The condition numbers are found to be independent of the number of subdomains for a certain polynomial order and stabilization paramter. Also, it is observed that the condition number bound is almost independent of the stabilization parameter based on the current tests. But it increases with the increasing polynomial orders.

The next group of tables demonstrate results for the second set of experiments in which ρ

Number of Subdomains	Iterations	Condition number
4×4	4	3.08
8×8	9	3.45
12×12	10	3.52
16×16	11	3.54
20×20	11	3.55

Table 4.9: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\tau \equiv 1$, $k = 2$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

Number of Subdomains	Iterations	Condition number
4×4	4	1.86
8×8	8	2.15
12×12	9	2.21
16×16	10	2.24
20×20	10	2.25

Table 4.10: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\tau \equiv \frac{1}{h}$, $k = 0$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

Number of Subdomains	Iterations	Condition number
4×4	4	2.28
8×8	8	2.65
12×12	9	2.73
16×16	10	2.76
20×20	10	2.77

Table 4.11: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\tau \equiv \frac{1}{h}$, $k = 1$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

Number of Subdomains	Iterations	Condition number
4×4	4	3.10
8×8	9	3.45
12×12	10	3.51
16×16	11	3.54
20×20	11	3.55

Table 4.12: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, $\tau \equiv \frac{1}{h}$, $k = 2$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

$\frac{H}{h}$	Iterations	Condition number
4	7	1.68
8	8	2.08
12	8	2.32
16	8	2.49
20	8	2.62

Table 4.13: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, $\tau \equiv 1$, $k = 0$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

$\frac{H}{h}$	Iterations	Condition number
4	8	2.32
8	8	2.74
12	9	2.99
16	9	3.16
20	9	3.30

Table 4.14: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, $\tau \equiv 1$, $k = 1$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

is in a checkboard pattern with $\rho = 1$ or $\rho = 1000$. Again, the size of the subdomain problem is fixed to be $\frac{H}{h} = 8$. Compared with results from the first set of experiments, the inhomogeneity of the coefficient ρ almost does not degrade the performance of the preconditioner.

Results from the third set of experiments are given in the last group of tables. In these cases, ρ is still in a checkboard pattern with $\rho = 1$ or $\rho = 1000$. But instead of fixing the size of the subdomain problems, we fix the subdomain partition to be 8×8, and allow the subdomain problem size to vary. The condition number is found to increase logarithmically

$\frac{H}{h}$	Iterations	Condition number
4	9	3.23
8	9	3.45
12	9	3.61
16	9	3.75
20	9	3.85

Table 4.15: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, $\tau \equiv 1$, $k = 2$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

$\frac{H}{h}$	Iterations	Condition number
4	8	1.76
8	8	2.15
12	8	2.39
16	8	2.57
20	8	2.70

Table 4.16: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, $\tau \equiv \frac{1}{h}$, $k = 0$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

$\frac{H}{h}$	Iterations	Condition number
4	8	2.23
8	8	2.65
12	9	2.90
16	9	3.07
20	9	3.21

Table 4.17: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, $\tau \equiv \frac{1}{h}$, $k = 1$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

$\frac{H}{h}$	Iterations	Condition number
4	9	3.25
8	9	3.45
12	9	3.60
16	9	3.73
20	9	3.83

Table 4.18: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, $\tau \equiv \frac{1}{h}$, $k = 2$, and ρ is in a checkerboard pattern with $\rho = 1$ or $\rho = 1000$.

with the subdomain problem size when other changing parameters are kept to be the same.

To conclude, we have carried out a series of experiments to obtain iteration counts and condition number estimates. The experimental results prove to be consistent with the theory. That is the condition number bound of the BDDC preconditioned system is of the form $C\left(1+\log\frac{H}{h}\right)^2$, where H and h are the diameters of the subdomains and elements, respectively, and C is almost independent of coefficients in the original equation, the stabilization parameter of the numerical scheme, but dependent on the orders of the approximating polynomial. Possible future work will be to explore the high order effects on C.

4.2 BDDC for Elliptic Problem with WG Method

4.2.1 Introduction

The weak Galerkin (WG) methods are a class of nonconforming finite element methods, which were first introduced for a second order elliptic problem in Wang and Ye (Wang & Ye, 2013). The idea of the WG is to introduce weak functions and their weak derivatives as distributions, which can be approximated by polynomials of different degrees. For second order elliptic problems, weak functions have the form of $v = \{v_0; v_b\}$, where v_0 is defined inside each element and v_b is defined on the boundary of the element. v_0 and v_b can both be approximated by polynomials. The gradient operator is approximated by a *weak gradient* operator, which is further approximated by polynomials. These weakly defined functions and derivatives make the WG methods highly flexible and these WG methods have been extended to different applications such as Darcy in Lin et al. (Lin et al., 2014), Stokes in (Wang & Ye, 2016), bi-harmonic in Mu et al. (Mu et al., 2014b), Maxwell in Mu et al. (Mu et al., 2015c), Helmholtz in Mu et al. (Mu et al., 2015a), and Brinkman equations in Mu et al. (Mu et al., 2014a). In Mu et al. (Mu et al., 2015b), the optimal order of polynomial spaces is studied to minimize the number of degrees of freedom in the computation.

The WG methods are closely related to the hybridizable discontinuous Galerkin (HDG)

methods, which were introduced by Cockburn and his collaborators in Cockburn et al. (Cockburn et al., 2009a). As most DG methods, the WG methods result in a large number of degrees of freedom and therefore require solving large linear systems with condition number deteriorating with the refinement of the mesh. Efficient fast solvers for the resulting linear system are necessary. However, so far there are relatively few fast solvers for the WG methods. Some multigrid methods, based on conforming finite element discretization, are studied in Chen et al. (Chen et al., 2015).

The BDDC algorithms, introduced by Dohrmann for second order elliptic problem in Dohrmann (Dohrmann, 2003), see also Mandel and Dohrmann (Mandel & Dohrmann, 2003), Mandel et al. (Mandel et al., 2005), are non-overlapping domain decomposition methods, which are similar to the balancing Neumann-Neumann (BNN) algorithms. In the BDDC algorithm, the coarse problems are given in terms of a set of primal constraints. An important advantage with such a coarse problem is that the Schur complements that arise in the computation will all be invertible. The BDDC algorithms have been extended to the second order elliptic problem with mixed and hybrid formulations in Tu (Tu, 2005, 2007a) and the Stokes problem in Li and Widlund (Li & Widlund, 2006a).

In this work, we apply the BDDC preconditioner directly to the system arising from the WG discretization and estimate the condition number of the resulting preconditioned operator using its spectral equivalence with that of a hybridized RT method, which have been studied in Tu (Tu & Wang, 2016).

4.2.2 An elliptic problem Setting and its WG discretization

Let $\Omega \subset R^n$ be a simply connected bounded polygon ($n = 2, 3$). Consider the following second order scalar elliptic problem with a Dirichlet boundary condition:

$$\begin{cases} -\nabla \cdot (\boldsymbol{a} \nabla u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{4.24}$$

where $\boldsymbol{a}(x) \in [L^\infty(\Omega)]^{n\times n}$ is a symmetric positive-definite matrix function, $f \in L^2(\Omega)$, and $g \in H^{1/2}(\partial\Omega)$. In particular, $\xi^T \boldsymbol{a}(x)\xi \geq \alpha \parallel \xi \parallel^2$, for $x \in \Omega$ a.e., and some positive constant α. Without loss of generality, we assume that $g = 0$. If Ω is convex or has a C^2 boundary, then equation (4.24), with sufficiently smooth coefficient $\boldsymbol{a}$, has a unique solution $u \in H^2(\Omega)$.

We approximate u by discontinuous finite element spaces. Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω, and K be the element in $\mathcal{T}_h$. For any $K \in \mathcal{T}_h$, we denote by h_K the diameter of K with $h \coloneqq \max_{K\in\mathcal{T}_h} h_K$. Define $\mathcal{F}_h$ be the set of edges/faces of elements $K \in \mathcal{T}_h$. $\mathcal{F}_h^i$ and $\mathcal{F}_h^\partial$ are subsets of $\mathcal{F}_h$, which consists of domain interior and boundary edges, respectively. Let $P^k(D)$ be the space of polynomials of degree $\leq k$ on D. Define the weak Galerkin finite element spaces associated with $\mathcal{T}_h$ as:

$$V_k = \{v = \{v_0, v_b\} : v_0 \in W_k,\ v_b \in M_{k-1}\},$$

where

$$W_k = \{w_h \in L^2(\Omega) : w_h \mid_K \in P_k(K), \quad \forall K \in \mathcal{T}_h\},$$
$$M_k = \{\mu_h \in L^2(\mathcal{F}_h) : \mu_h \mid_e \in P_k(e), \quad \forall e \in \mathcal{F}_h\}.$$

A function $v \in V_k$ has a single value v_b on each $e \in \mathcal{F}_h$.

Let

$$V_k^0 = \{v \in V_k : v_b = 0 \text{ on } \partial\Omega\}.$$

Denote by $\nabla_{w,k-1}$ the discrete weak gradient operator on the finite element space V_k. It is defined as follows: for $v = \{v_0, v_b\} \in V_k$, on each element $K \in \mathcal{T}_h$, $\nabla_{w,k-1} v \mid_K \in [P_{k-1}(K)]^n$ is the unique solution of the following equation

$$\left(\nabla_{w,k-1} v \mid_K, \mathbf{q}\right)_K = -\left(v_{0,K}, \nabla\cdot\mathbf{q}\right)_K + \left\langle v_{b,K}, \mathbf{q}\cdot\mathbf{n}\right\rangle_{\partial K}, \qquad \forall \mathbf{q} \in [P_{k-1}(K)]^n,$$

where $v_{0,K}$ and $v_{b,K}$ are the restrictions of v_0 and v_b to K, respectively. To simplify the

notation, we will drop the subscript $k-1$ in the discrete weak gradient operator $\nabla_{w,k-1}$. We use the common notations for L^2-inner products. Write $(u,v)_D = \int_D uvdx$ whenever D is a domain of $\mathbb{R}^n$ and $\langle u,v\rangle_D = \int_D uvdx$ whenever D is an $(n-1)$-dimensional domain. In particular, we have $(u,w)_K = \int_K uwdx$ and $\langle u,w\rangle_{\partial K} = \int_{\partial K} uwds$. Define $(v,w)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}(v,w)_K$ and $\langle v,w\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}\langle v,w\rangle_{\partial K}$.

The discrete problem resulting from the WG discretization can then be written as: find $u_h = (u_0, u_b) \in V_k^0$ such that

$$a(u_h, v_h) + s(u_h, v_h) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in V_k^0, \tag{4.25}$$

where

$$\begin{aligned} a(u_h, v_h) &= (\nabla_w u_h, \nabla_w v_h)_{\mathcal{T}_h}, \\ s(u_h, v_h) &= \sum_{K\in\mathcal{T}_h} h_K^{-1} \langle Q_b u_0 - u_b, Q_b v_0 - v_b\rangle_{\partial K}. \end{aligned}$$

Herein, Q_b denotes the L^2 projection from $L^2(e)$ to $P_{k-1}(e)$, for $e \in \partial K$. In Mu et al. (Mu et al., 2015b), (4.25) is proved to have a unique solution and the approximation properties of the WG methods are also studied.

Given a $u_h \in V_k$, let $\boldsymbol{q}_K = \nabla_w u_h\,|_K$ and write (4.25) as a system of $\boldsymbol{q}$, u_0, u_b, which is similar to the linear system resulting from the HDG discretization with the local stabilization parameter h_K^{-1}. Given the value of u_b on ∂K, $\boldsymbol{q}$ and u_0 can be uniquely determined in K, see Cockburn et al. (Cockburn et al., 2009a). Therefore, by eliminating $\nabla_w u\,|_K$ and u_0 locally in each element, (4.25) can be reduced to a system in u_b only

$$Au_b = b, \tag{4.26}$$

where b is the corresponding right-hand-side function.

In the next section, we will develop a BDDC algorithm to solve the system in (4.26) for

u_b. To make the notation simple, we will denote u_b by λ and the finite element space for u_b by $\Lambda = \{\mu \in M_{k-1} : \mu \mid_e = 0 \; \forall e \in \partial\Omega\}$.

4.2.3 The BDDC Algorithms

We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such triangles forming an individual subdomain is uniformly bounded. We also assume $\boldsymbol{a}(x)$, the coefficient of (4.24), is constant in each subdomain. We reduce the global problem (4.26) to a subdomain interface problem. Let Γ be the interface between subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = \left(\cup_{i \neq j} \partial\Omega_{i,h} \cap \partial\Omega_{j,h}\right) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$.

We can decompose Λ into the subdomain interior and interface parts as

$$\Lambda = \bigoplus_{i=1}^{N} \Lambda_I^{(i)} \bigoplus \widehat{\Lambda}_\Gamma.$$

We denote the subdomain interface numerical trace space of Ω_i by $\Lambda_\Gamma^{(i)}$, and the associate product space by $\Lambda_\Gamma = \prod_{i=1}^{N} \widehat{\Lambda}_\Gamma^{(i)}$. $R_\Gamma^{(i)}$ is the operator which maps functions in the continuous interface numerical trace space $\widehat{\Lambda}_\Gamma$ to their subdomain components in the space $\Lambda_\Gamma^{(i)}$. The direct sum of the $R_\Gamma^{(i)}$ is denoted by R_Γ. We can eliminate the subdomain interior variables $\lambda_I^{(i)}$ in each subdomain independently and define the subdomain Schur complement $S_\Gamma^{(i)}$ by: given $\lambda_\Gamma^{(i)} \in \Lambda_\Gamma^{(i)}$, determine $S_\Gamma^{(i)} \lambda_\Gamma^{(i)}$ such that

$$\begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \lambda_I^{(i)} \\ \lambda_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ S_\Gamma^{(i)} \lambda_\Gamma \end{bmatrix}. \tag{4.27}$$

The global interface problem is assembled from the subdomain interface problems, and

can be written as: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$\widehat{S}_\Gamma \lambda_\Gamma = b_\Gamma, \tag{4.28}$$

where $b_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} b_\Gamma^{(i)}$, and $\widehat{S}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}$. Thus, $\widehat{S}_\Gamma$ is a symmetric, positive definite operator defined on the interface space $\widehat{\Lambda}_\Gamma$. We will propose a BDDC preconditioner for solving (4.28) with a preconditioned conjugate gradient method.

In order to introduce the BDDC preconditioner, we first introduce a partially assembled interface space $\widetilde{\Lambda}_\Gamma$ by

$$\widetilde{\Lambda}_\Gamma = \widehat{\Lambda}_\Pi \bigoplus \Lambda_\Delta = \widehat{\Lambda}_\Pi \bigoplus \left(\prod_{i=1}^N \Lambda_\Delta^{(i)} \right).$$

Here, $\widehat{\Lambda}_\Pi$ is the coarse level, primal interface space which is spanned by subdomain interface edge basis functions with constant values at the nodes of the edge for two dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit, see (Li & Widlund, 2006b) and (Klawonn & Widlund, 2006). The new variables are called the primal unknowns. The space Λ_Δ is the direct sum of the $\Lambda_\Delta^{(i)}$, which are spanned by the remaining interface degrees of freedom with a zero average over each edge. In the space $\widetilde{\Lambda}_\Gamma$, we relax most continuity constraints across the interface but retain the continuity at the primal unknowns, which makes all the linear systems nonsingular.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\Lambda}_\Gamma$ to the components $\Lambda_\Gamma^{(i)}$ of the subdomain Ω_i. $R_\Delta^{(i)}$ maps the functions from $\widehat{\Lambda}_\Gamma$ to $\Lambda_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi}$ is a restriction operator from $\widehat{\Lambda}_\Gamma$ to its subspace $\widehat{\Lambda}_\Pi$. $\overline{R}_\Gamma : \widetilde{\Lambda}_\Gamma \to \Lambda_\Gamma$ is the direct sum of the $\overline{R}_\Gamma^{(i)}$ and $\widetilde{R}_\Gamma : \widehat{\Lambda}_\Gamma \to \widetilde{\Lambda}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define a positive scaling factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. We note that $\delta_i^\dagger(x)$ is constant on each edge, since we assume that the $\rho_i(x)$ is constant in each subdomain, and the nodes on each edge are shared by the same subdomains. Multiplying each row of $R_\Delta^{(i)}$, with the scaling factor $\delta_i^\dagger(x)$, gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_{D,\Delta}^{(i)}$.

The partially assembled interface Schur complement is defined by $\widetilde{S}_\Gamma = \overline{R}_\Gamma^T diag(S_\Gamma^{(i)})\overline{R}_\Gamma$, and the preconditioned BDDC operator is then of the form: find $\lambda_\Gamma \in \widehat{\Lambda}_\Gamma$, such that

$$\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \widehat{S}_\Gamma \lambda_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} b_\Gamma. \tag{4.29}$$

The system above can be solved by the preconditioned conjugate gradient method.

4.2.4 Auxiliary Results

Denote by C a generic constant independent of mesh size h. Its value may differ at different occurrences. We prove the spectral equivalence of A, defined in (4.26), and the triple-bar norm defined in(Gopalakrishnan, 2003) as below:

$$|||\lambda|||_{h,D}^* = \left(\sum_{K \in \mathcal{T}_h, K \subseteq \overline{D}} \frac{1}{h} \|\lambda - m_K(\lambda)\|_{\partial K}^2 \right)^{1/2}, \tag{4.30}$$

where $m_K(\lambda) = \frac{1}{|\partial K|} \int_{\partial K} \lambda ds$. Recall that this triple-bar norm was first introduced in (4.18).

Define the local lifting operators $\mathcal{Q}(\cdot)$ and $\mathcal{U}(\cdot)$ for the weak Galerkin (WG) method as below:

$$(\mathcal{Q}\mu, \boldsymbol{r})_K + (\mathcal{U}\mu, \nabla \cdot \boldsymbol{r})_K = \langle \mu, \boldsymbol{r} \cdot \boldsymbol{n} \rangle_{\partial K} \quad \text{for all } \boldsymbol{r} \in [P_{k-1}(K)]^n, \tag{4.31a}$$

$$-(w, \nabla \cdot \mathcal{Q}\mu)_K + \langle h^{-1}(Q_b \mathcal{U}\mu - \mu), Q_b w \rangle_{\partial K} = 0 \quad \text{for all } w \in P_k(K). \tag{4.31b}$$

Note that the connection between (4.31) and (4.25) can be revealed for the case $f = 0$ as

follows:

$$\nabla_w u_h = \mathcal{Q}\lambda; \qquad u_h = (\mathcal{U}\lambda, \lambda),$$

where $(\mathcal{U}\lambda, \lambda)$ is an ordered pair.

Following the work in (Gopalakrishnan, 2003),we prove the equivalence between the triple-bar norms obtained from the WG bilinear form

$$|||\lambda|||^2 = a_h(\lambda,\lambda) = (\mathcal{Q}\lambda, \mathcal{Q}\lambda)_{\mathcal{T}_h} + \langle h^{-1}(Q_b\mathcal{U}\lambda - \lambda), Q_b\mathcal{U}\lambda - \lambda\rangle_{\partial\mathcal{T}_h}$$

and $|||\cdot|||^*$ as defined in (4.30). To denote the triple-bar norm defined over an element K, we add a subscript K to it.

Lemma 4.2.1. *The function $|||\lambda|||_K$ is zero on $K \in \mathcal{T}_h$ if and only if λ is constant on ∂K.*

Proof. Assume that $|||\lambda|||_K = 0$ on K. It follows that

$$0 = (\nabla_w u, \nabla_w u) + h^{-1}\langle Q_b\mathcal{U}\lambda - \lambda, Q_b\mathcal{U}\lambda - \lambda\rangle_{\partial K},$$

where $u = \{\mathcal{U}\lambda, \lambda\}$, and $\nabla_w u = \mathcal{Q}\lambda$. This implies that $\nabla_w u = 0$ on element K and $Q_b\mathcal{U}\lambda = \lambda$ on ∂K. Further, we have from the definition of the discrete weak gradient operator or the lifting operator $\mathcal{Q}$ given in (4.31b) that for any $\tau \in [P_{k-1}(K)]^n$,

$$\begin{aligned}
0 &= (\nabla_w u, \tau)_K \\
&= -(\mathcal{U}\lambda, \nabla\cdot\tau)_K + \langle\lambda, \tau\cdot n\rangle_{\partial K} \\
&= (\nabla\mathcal{U}\lambda, \tau)_K - \langle\mathcal{U}\lambda - \lambda, \tau\cdot n\rangle_{\partial K} \\
&= (\nabla\mathcal{U}\lambda, \tau)_K - \langle Q_b\mathcal{U}\lambda - \lambda, \tau\cdot n\rangle_{\partial K} \\
&= (\nabla\mathcal{U}\lambda, \tau)_K.
\end{aligned}$$

Let $\tau = \nabla\mathcal{U}\lambda$. Then we have $\nabla\mathcal{U}\lambda = 0$ on K. It follows that $\mathcal{U}\lambda = const.$ on K. Thus, $Q_b\mathcal{U}\lambda = const.$ on ∂K. Since $Q_b\mathcal{U}\lambda = \lambda$ on ∂K, we have $\lambda = const.$ Note that similar

argument as above was provided in Wang & Ye (2016) to prove the faithfulness of the norm $|||\cdot|||$.

Conversely, assume λ is constant on ∂K. Substituting the ordered pair (r, w) in (4.31) with $(\mathcal{Q}\lambda, \mathcal{U}\lambda)$ and adding up, we obtain

$$|||\lambda|||_K^2 = \langle \lambda,\, \mathcal{Q}\lambda \cdot n\rangle_{\partial K} - h^{-1}\langle Q_b\mathcal{U}\lambda - \lambda,\, \lambda\rangle_{\partial K}.$$

Let $w = \lambda$ be the test function in (4.31b). Since λ is constant, $\lambda = Q_b\lambda$. It follows from (4.31b) that

$$-\langle \lambda,\, \mathcal{Q}\lambda \cdot n\rangle_{\partial K} + h^{-1}\langle Q_b\mathcal{U}\lambda - \lambda,\, \lambda\rangle_{\partial K} = 0.$$

Therefore, $|||\lambda|||_K = 0$.

□

Lemma 4.2.2. *Let* $M_h = \{v_b : v = \{v_0, v_b\} \in V_k^0\}$. *For all* $\lambda \in M_h$,

$$c|||\lambda|||_h^{*,2} \leq |||\lambda|||^2 \leq C|||\lambda|||_h^{*,2}.$$

Proof. First, we prove the lower bound. By Lemma 4.2.1, $|||\lambda|||_K = 0$ implies that λ is constant on ∂K. Similarly as in Gopalakrishnan (2003), by a scaling argument, it can be shown that

$$|||\lambda|||_K \geq \frac{c}{|\partial K|^{1/2}} \inf_{\kappa\in\mathbb{R}} \|\lambda - \kappa\|_{\partial K} = \frac{c}{|\partial K|^{1/2}} \|\lambda - m_K(\lambda)\|_{\partial K} = c|||\lambda|||_{h,K}^*,$$

for some constant c independent of λ.

Next, we prove the upper bound. Let $r = \mathcal{Q}\lambda$, and $w = \mathcal{U}\lambda$. Plugging the ordered pair (r, w)

into (4.31), and adding up, we obtain

$$\begin{aligned}
|||\lambda|||_K^2 &= \langle \lambda,\, \mathcal{Q}\lambda \cdot n \rangle_{\partial K} - h^{-1} \langle Q_b \mathcal{U}\lambda - \lambda,\, \lambda \rangle_{\partial K} \\
&= \langle \lambda,\, \mathcal{Q}\lambda \cdot n - h^{-1}(Q_b \mathcal{U}\lambda - \lambda) \rangle_{\partial K} \\
&= \langle \lambda - m_K(\lambda),\, \mathcal{Q}\lambda \cdot n - h^{-1}(Q_b \mathcal{U}\lambda - \lambda) \rangle_{\partial K} \\
&\leq \frac{C}{|\partial K|^{1/2}} \|\lambda - m_K(\lambda)\|_{\partial K} |||\lambda|||_K \\
&= C|||\lambda|||_{h,K}^* |||\lambda|||_K,
\end{aligned}$$

where we have used (4.31b) for the third equality, the trace inequality (1.5.3) and inverse inequality (1.5.4) for the second-to-last inequality. It follows that

$$c|||\lambda|||_{h,K}^{*,2} \leq |||\lambda|||_K^2 \leq C|||\lambda|||_{h,K}^{*,2}.$$

Summing up over all elements in $\mathcal{T}_h$, we obtain

$$c|||\lambda|||_h^{*,2} \leq |||\lambda|||^2 \leq C|||\lambda|||_h^{*,2}.$$

□

4.2.5 Condition Number Bound

We define the interface averaging operator E_D, by

$$E_D = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T,$$

which computes a weighted average across the subdomain interface Γ and then distributes the averages to the degrees of freedom on the boundary of the subdomains.

Based on the equivalence of norms in Lemma 4.2.2, similar to the proof of Lemma 4.1.5 or (Tu & Wang, 2016, Lemma 5), we can obtain that the interface averaging operator E_D satisfies the following bound:

Lemma 4.2.3. *For any $\lambda_\Gamma \in \widetilde{\Lambda}_\Gamma$,*

$$|E_D\lambda_\Gamma|^2_{\widetilde{S}_\Gamma} \leq C\left(1+\log\frac{H}{h}\right)^2 |\lambda_\Gamma|^2_{\widetilde{S}_\Gamma},$$

where C is a positive constant independent of H, h, and the coefficient of (4.1).

As in the proof of (Li & Widlund, 2006a, Theorem 1), (Tu & Wang, 2016, Theorem 1), and Theorem 4.1.6, using Lemma 4.2.3, we can obtain

Theorem 4.2.4. *The condition number of the preconditioned operator $M^{-1}\widehat{S}_\Gamma$ is bounded by $C(1+\log\frac{H}{h})^2$, where C is a constant which is independent of h, H, and the coefficients $\boldsymbol{a}$ of (4.1).*

4.2.6 Numerical Experiments

We have applied our BDDC algorithms to the model problem (4.1), where $\Omega = [0,1]^2$. We decompose the unit square into $N \times N$ subdomains with the sidelength $H = 1/N$. Equation (4.1) is discretized, in each subdomain, by the kth-order WG method with a element diameter h. The preconditioned conjugate gradient iteration is stopped when the relative l_2-norm of the residual has been reduced by a factor of 10^6.

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. In the first set of experiments, we take the coefficient $\boldsymbol{a} \equiv 1$. In the second set of experiments, we take the coefficient $\boldsymbol{a} = 1$ in half of the subdomains and $\boldsymbol{a} = 1000$ in the neighboring subdomains, in a checkerboard pattern. All the experimental results are fully consistent with our theory.

Table 4.19: Performance with $H/h = 8/\#sub = 64$

		$\rho = 1$				ρ checkboard pattern			
		$k = 1$		$k = 2$		$k = 1$		$k = 2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	2.22	6	3.50	7	1.80	5	2.37	5
	8×8	2.45	13	3.85	16	2.08	9	2.76	10
	16×16	2.45	14	3.86	17	2.16	14	2.87	15
	24×24	2.46	14	3.87	17	2.17	15	2.89	15
	32×32	2.46	14	3.87	17	2.18	15	2.90	16
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	1.78	11	2.90	14	1.67	9	2.33	10
	8	2.45	13	3.86	16	2.08	9	2.76	10
	16	3.29	15	4.95	18	2.49	10	3.18	10
	24	3.85	17	5.67	18	2.74	10	3.43	11
	32	4.28	17	6.21	19	2.91	10	3.60	11

Chapter 5

BDDC Algorithms for Stokes flow

The main results from the first section are included in a manuscript in preparation(Tu & Wang, 2017a); those from the second section are included in another manuscript in preparation(Tu & Wang, 2017b).

5.1 BDDC for Stokes Problem with HDG Method

5.1.1 Introduction

Nonoverlapping domain decomposition methods have been studied well for solving saddle-point problems; see, e.g., (Pavarino & Widlund, 2002; Li, 2005; Goldfeld et al., 2003; Dohrmann, 2007b; Li & Widlund, 2006a; Tu, 2007a, 2005; Li, 2005; Li & Tu, 2013; Tu & Li, 2014, 2013, 2015; Pavarino & Scacchi, 2016). In many of these works, the original saddle point problems are reduced to positive definite problems in a subspace called the benign subspace and the conjugate gradient (CG) methods are used to solve the system. In order to make all CG iterates in the benign subspace, one has to deal with the so-called no-net-flux constraints across subdomain boundaries, which often lead to large coarse level problems. The no-net-flux constraints can be complicated for the incompressible Stokes equations with standard finite element discretization, especially in three dimensions (Li & Widlund, 2006a).

Moreover, the large coarse level problem will be a bottleneck in large scale parallel computations, and some inexact solvers in the algorithms are needed to reduce its impact, cf. (Tu, 2007d,c,b; Klawonn & Rheinbach, 2007; Dohrmann, 2007a; Kim & Tu, 2009; Tu, 2011).

The Balancing Domain Decomposition by Constraints (BDDC) algorithms, introduced by Dohrmann for second order elliptic problem in (Dohrmann, 2003) and analyzed in (Mandel & Dohrmann, 2003; Mandel et al., 2005), are nonoverlapping domain decomposition methods, which are similar to the balancing Neumann-Neumann (BNN) algorithms. In the BDDC algorithm, the coarse problems are given in terms of a set of primal constraints. An important advantage with such a coarse problem is that the Schur complements that arise in the computation will all be invertible. The BDDC algorithms have been extended to the second order elliptic problem with mixed and hybrid formulations (Tu, 2005, 2007a), with isogeometric collocation methods and spectral elements (Beirão da Veiga et al., 2013, 2014; Canute et al., 2014), with staggered discontinuous Galerkin methods (Kim et al., 2014), with hybridizable discontinuous Galerkin (HDG) discretization (Tu & Wang, 2016), and the incompressible Stokes problem with conforming finite element discretization (Li & Widlund, 2006a).

In this work, the BDDC algorithm is developed for the incompressible Stokes equation with HDG discretization. General HDG methods were introduced by Cockburn and his collaborators in (Cockburn et al., 2009a) and the specific HDG method we consider here is often called LDG-H method, which is constructed by using the local discontinuous Galerkin method on each element. One distinct feature of the HDG method, applied to a scalar elliptic problem, is that the only global coupled degrees of freedom are a scalar variable, called "numerical traces". Therefore the resulting global system from the HDG is much smaller than other traditional DG methods. The HDG discretization for incompressible Stokes flow has been introduced in (Nguyen et al., 2010) and analyzed in (Cockburn et al., 2011). The main features of this approach is that it reduces the globally coupled unknowns to the numerical trace of the velocity and the mean of the pressure on the element boundaries.

The size of the reduced saddle point problem is significantly smaller compared to the original one. In (Nguyen et al., 2010), the reduced saddle point problem is solved by an augmented Lagrange approach. An additional time dependent problem is introduced and solved by a backward-Euler method. Here, we solve the reduced saddle point problem directly using the BDDC methods. Similar to the earlier domain decomposition works on saddle point problems (Pavarino & Widlund, 2002; Li, 2005; Li & Widlund, 2006a; Tu, 2005), we reduce the saddle point problem to a positive definite problem in a benign subspace and therefore the CG method can be used to solve the resulting system. Compare to the standard finite element discretization, the HDG discretization has discontinuous pressure basis functions, which make the application of the BDDC algorithm much easier, see (Li & Tu, 2013; Tu & Li, 2014, 2013, 2015). Moreover, the complicated no-net-flux condition, which is needed to make sure all CG iterates in the benign subspace, can be ensured by edge and face average constraints for each velocity component in two and three dimensions, respectively. These constraints are the same as those for the elliptic problems with the HDG discretizations (Tu & Wang, 2016). This fact makes the BDDC algorithm much simpler than those with standard finite element discretizations.

Following a similar approach used in (Wang & Ye, 2016) for a weak Galerkin finite element method for the Stokes equation, we prove the inf-sup stability of one class of the HDG methods discussed in (Nguyen et al., 2010). Based on this result, we establish the relation between the Stokes and Harmonic extensions with this class of the HDG discretization. The relation is important in the condition number estimate and the similar relation for the standard finite element method is provided in (Bramble & Pasciak, 1989). Combining all these results and the condition number bound for the elliptic problem in (Tu & Wang, 2016), we obtain the condition number estimate of the BDDC preconditioned Stokes operator.

5.1.2 A Stokes problem and HDG Discretization

We consider the following Stokes problem on a bounded polygonal domain Ω, in two or three dimensions, with a Dirichlet boundary condition:

$$\begin{cases} -\triangle \mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = g & \text{on} \quad \partial\Omega, \end{cases} \tag{5.1}$$

where $\mathbf{f} \in \mathbf{L^2}(\mathbf{\Omega})$ and $g \in H^{1/2}(\partial\Omega)$. Without loss of generality, we assume that $g = 0$. The solution of (5.1) is unique for the pressure p up to a constant. Here we will look for the solution with the pressure p having a zero average over the domain Ω.

We follow the approach in (Nguyen et al., 2010; Cockburn et al., 2011) and introduce the HDG method for the velocity-pressure-gradient formulation of the Stokes equation as follows:

$$\begin{cases} \mathbf{L} - \nabla \mathbf{u} = 0 & \text{in} \quad \Omega, \\ -\nabla \cdot \mathbf{L} + \nabla p = \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{5.2}$$

We will approximate $\mathbf{L}$, $\mathbf{u}$, and p by introducing discontinuous finite element spaces. Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω with characteristic element size h and the element in $\mathcal{T}_h$ denoted by κ. Define $\mathcal{E}$ to be the union of edges of elements κ. $\mathcal{E}_i$ and $\mathcal{E}_\partial$ are the sets of the domain interior and boundary edges, respectively.

Let $P_k(D)$ be the space of polynomials of order at most k on D. We set $\mathbf{P}_k(D) = [P_k(D)]^n$ ($n = 2$ and 3 for two and three dimensions, respectively) and $\mathcal{P}_k(D) = [P_k(D)]^{n \times n}$.

For each element κ, we find $(L_h, \mathbf{u}_h, p_h) \in (\mathcal{P}_k(\kappa), \mathbf{P}_k(\kappa), P_k(\kappa))$ such that for all $\kappa \in \mathcal{T}_h$

$$\begin{cases} (\mathbf{L}_h, \mathbf{G}_h)_\kappa + (\mathbf{u}_h, \nabla \cdot \mathbf{G}_h)_\kappa - \langle \widehat{\mathbf{u}}_h, \mathbf{G}_h \mathbf{n} \rangle_{\partial\kappa} &= 0, & \forall\ \mathbf{G}_h \in \mathcal{P}_k(\kappa), \\ (\mathbf{L}_h, \nabla \mathbf{v}_h)_\kappa - (p_h, \nabla \cdot \mathbf{v}_h)_\kappa - \left\langle \widehat{\mathbf{L}}_h \mathbf{n} - \widehat{p}_h \mathbf{n}, \mathbf{v}_h \right\rangle_{\partial\kappa} &= (\mathbf{f}, \mathbf{v}_h)_\kappa, & \forall\ \mathbf{v}_h \in \mathbf{P}_k(\kappa), \\ -(\mathbf{u}_h, \nabla q_h)_\kappa + \langle \widehat{\mathbf{u}}_h \cdot \mathbf{n}, q_h \rangle_{\partial\kappa} &= 0, & \forall\ q_h \in P_k(\kappa), \end{cases} \tag{5.3}$$

where $(\cdot,\cdot)_\kappa$ and $\langle \cdot,\cdot \rangle_{\partial\kappa}$ denote L^2-inner product of functions or vector-valued functions in κ and $\partial\kappa$, respectively. $\widehat{\mathbf{L}}_h$, $\widehat{\mathbf{u}}_h$, and $\widehat{p}_h$ are the numerical traces which approximate $\mathbf{L}_h$, $\mathbf{u}_h$ and p_h on $\partial\kappa$ respectively.

Define the following finite element spaces:

$$\begin{aligned} \mathbf{G}_k &= \{\mathbf{G}_h \in [L^2(\Omega)]^(n \times n) : \mathbf{G}_h|_\kappa \in \mathcal{P}_k(\kappa) \quad \forall \kappa \in \Omega\}, \\ \mathbf{V}_k &= \{\mathbf{v}_h \in [L^2(\Omega)]^n : \mathbf{v}_h|_\kappa \in \mathbf{P}_k(\kappa) \quad \forall \kappa \in \Omega\}, \\ W_k &= \{p_h \in L^2(\Omega) : p_h|_\kappa \in P_k(\kappa), \quad \int_\Omega p_h = 0, \quad \forall \kappa \in \Omega\}, \\ \mathbf{M}_k &= \{\boldsymbol{\mu}_h \in [L^2(\mathcal{E})]^n : \boldsymbol{\mu}_h|_e \in \mathbf{P}_k(e) \quad \forall e \in \mathcal{E}\}. \end{aligned}$$

Let $\boldsymbol{\Lambda}_k = \{\boldsymbol{\mu}_h \in \mathbf{M}_k : \boldsymbol{\mu}_h|_e = 0\ \forall e \in \partial\Omega\}$. To make our notations simple, we drop the subscript k from now on.

Let $\boldsymbol{\lambda}_h \in \boldsymbol{\Lambda}$ and the numerical trace $\widehat{\mathbf{u}}_h = \boldsymbol{\lambda}_h$. The numerical flux $\widehat{\mathbf{L}}_h \mathbf{n} - \widehat{p}_h \mathbf{n}$ is more complicated and see (Nguyen et al., 2010; Cockburn et al., 2011) for more general discussion. In this work, we consider the following numerical trace:

$$\widehat{\mathbf{L}}_h \mathbf{n} - \widehat{p}_h \mathbf{n} = \mathbf{L}_h \mathbf{n} - p_h \mathbf{n} - \tau_\kappa (\mathbf{u}_h - \boldsymbol{\lambda}_h), \quad \text{on } \partial\kappa, \tag{5.4}$$

where τ_κ is a local stabilization parameter, see (Nguyen et al., 2010; Cockburn et al., 2011) for details.

With the definitions of numerical trace $\widehat{\mathbf{u}}_h = \boldsymbol{\lambda}_h$ and the numerical flux $\widehat{\mathbf{L}}_h \mathbf{n} - \widehat{p}_h \mathbf{n}$, this discrete problem resulting from HDG discretization can be written as: to find $(\mathbf{L}_h, \mathbf{u}_h, p_h, \boldsymbol{\lambda}_h) \in$

$(\mathbf{G},\mathbf{V},W,\mathbf{\Lambda})$ such that for all $(\mathbf{G}_h,\boldsymbol{v}_h,q_h,\boldsymbol{\mu}_h)\in(\mathbf{G},\mathbf{V},W,\mathbf{\Lambda})$

$$\begin{cases} (\mathbf{L}_h,\mathbf{G}_h)_{\mathcal{T}_h}+(\mathbf{u}_h,\nabla\cdot\mathbf{G}_h)_{\mathcal{T}_h}-\langle\boldsymbol{\lambda}_h,\mathbf{G}_h\mathbf{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\ (\mathbf{L}_h,\nabla\mathbf{v}_h)_{\mathcal{T}_h}-(p_h,\nabla\cdot\mathbf{v}_h)_{\mathcal{T}_h}-\langle\mathbf{L}_h\mathbf{n}-p_h\mathbf{n}-\tau_\kappa(\mathbf{u}_h-\boldsymbol{\lambda}_h),\mathbf{v}_h\rangle_{\partial\mathcal{T}_h} &= (\mathbf{f},\mathbf{v}_h)_{\mathcal{T}_h},\\ -(\mathbf{u}_h,\nabla q_h)_{\mathcal{T}_h}+\langle\boldsymbol{\lambda}_h\cdot\mathbf{n},q_h\rangle_{\partial\mathcal{T}_h} &= 0,\\ -\langle\mathbf{L}_h\mathbf{n}-p_h\mathbf{n}-\tau_\kappa(\mathbf{u}_h-\boldsymbol{\lambda}_h),\boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h} &= 0. \end{cases} \tag{5.5}$$

Define

$$\begin{aligned} &A_{\mathbf{LL}}:\mathbf{G}\to\mathbf{G},\quad A_{\mathbf{uL}}:\mathbf{G}\to\mathbf{V},\quad A_{\boldsymbol{\lambda}\mathbf{L}}:\mathbf{G}\to\mathbf{\Lambda},\quad A_{\mathbf{uu}}:\mathbf{V}\to\mathbf{V},\\ &A_{\boldsymbol{\lambda}\mathbf{u}}:\mathbf{V}\to\mathbf{\Lambda},\quad B_{p\mathbf{u}}:\mathbf{V}\to W,\quad A_{\boldsymbol{\lambda}\boldsymbol{\lambda}}:\mathbf{\Lambda}\to\mathbf{\Lambda},\quad B_{p\boldsymbol{\lambda}}:\mathbf{\Lambda}\to W, \end{aligned} \tag{5.6}$$

as

$$(A_{\mathbf{LL}}\mathbf{L}_h,\mathbf{G}_h)=-(\mathbf{L}_h,\mathbf{G}_h)_{\mathcal{T}_h},\quad (A_{\mathbf{uL}}\mathbf{L}_h,\mathbf{v}_h)=-(\mathbf{L}_h,\nabla\mathbf{v}_h)_{\mathcal{T}_h},$$

$$(A_{\boldsymbol{\lambda}\mathbf{L}}\mathbf{L}_h,\boldsymbol{\mu}_h)=\langle\mathbf{L}_h\mathbf{n},\boldsymbol{\mu}_h\rangle_{\partial\mathcal{T}_h};\quad (A_{\mathbf{uu}}\mathbf{u}_h,\mathbf{v}_h)=\tau_\kappa\langle\mathbf{u}_h,\mathbf{v}_h\rangle_{\partial\mathcal{T}_h}$$

$$(A_{\boldsymbol{\lambda}\mathbf{u}}\mathbf{u}_h,\boldsymbol{\mu}_h)=-\tau_\kappa<\mathbf{u}_h,\boldsymbol{\mu}_h>_{\partial\mathcal{T}_h},\quad (A_{\boldsymbol{\lambda}\boldsymbol{\lambda}}\boldsymbol{\lambda}_h,\boldsymbol{\mu}_h)=\tau_\kappa<\boldsymbol{\lambda}_h,\boldsymbol{\mu}_h>_{\partial\mathcal{T}_h},$$

$$(B_{p\mathbf{u}}\mathbf{v}_h,p_h)=(\mathbf{v}_h,\nabla p_h)_{\mathcal{T}_h},\quad (B_{p\boldsymbol{\lambda}}\boldsymbol{\lambda}_h,p_h)=-<\boldsymbol{\lambda}_h\cdot\boldsymbol{n},p_h>_{\partial\mathcal{T}_h},$$

for all $\mathbf{L}_h,\mathbf{G}_h\in\mathbf{G}$, $\mathbf{u}_h,\mathbf{v}_h\in\mathbf{V}$, $p_h,q_h\in W$, and $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathbf{\Lambda}$.

Correspondingly, the matrix form of (5.5) is

$$\begin{bmatrix} A_{\mathbf{LL}} & A_{\mathbf{uL}}^T & A_{\boldsymbol{\lambda}\mathbf{L}}^T & 0\\ A_{\mathbf{uL}} & A_{\mathbf{uu}} & A_{\boldsymbol{\lambda}\mathbf{u}}^T & B_{p\mathbf{u}}^T\\ A_{\boldsymbol{\lambda}\mathbf{L}} & A_{\boldsymbol{\lambda}\mathbf{u}} & A_{\boldsymbol{\lambda}\boldsymbol{\lambda}} & B_{p\boldsymbol{\lambda}}^T\\ 0 & B_{p\mathbf{u}} & B_{p\boldsymbol{\lambda}} & 0 \end{bmatrix}\begin{bmatrix}\mathbf{L}\\ \mathbf{u}\\ \boldsymbol{\lambda}\\ p\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ F_h\\ \mathbf{0}\\ 0\end{bmatrix}, \tag{5.7}$$

where we use $\mathbf{L}$, $\mathbf{u}$, $\boldsymbol{\lambda}$, and p to denote the unknowns associated with $\mathbf{L}_h$, $\mathbf{u}_h$, $\boldsymbol{\lambda}_h$, and p_h,

respectively.

Let

$$A_a = \begin{bmatrix} A_{\mathbf{LL}} & A_{\mathbf{uL}}^T & A_{\boldsymbol{\lambda}\mathbf{L}}^T \\ A_{\mathbf{uL}} & A_{\mathbf{uu}} & A_{\boldsymbol{\lambda}\mathbf{u}}^T \\ A_{\boldsymbol{\lambda}\mathbf{L}} & A_{\boldsymbol{\lambda}\mathbf{u}} & A_{\boldsymbol{\lambda}\boldsymbol{\lambda}} \end{bmatrix}, \quad B_a^T = \begin{bmatrix} 0 \\ B_{p\mathbf{u}}^T \\ B_{p\boldsymbol{\lambda}}^T \end{bmatrix}, \quad u_a = \begin{bmatrix} \mathbf{L} \\ \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix}, \quad \text{and } F_a = \begin{bmatrix} \mathbf{0} \\ F_h \\ \mathbf{0} \end{bmatrix}, \tag{5.8}$$

the global problem (5.7) can be written as the following saddle point problem

$$\begin{bmatrix} A_a & B_a^T \\ B_a & 0 \end{bmatrix} \begin{bmatrix} u_a \\ p \end{bmatrix} = \begin{bmatrix} F_a \\ 0 \end{bmatrix}, \tag{5.9}$$

where A_a corresponds to the same HDG discretization for elliptic problem as discussed in (Tu & Wang, 2016).

In each κ, we decompose the pressure degrees of freedom p to the element average pressure p_{0e} and the rest called the element interior pressure p_i and let $W = W_i \oplus W_{0e}$, correspondingly. We can rewrite (5.7) as

$$\begin{bmatrix} A_{\mathbf{LL}} & A_{\mathbf{uL}}^T & 0 & A_{\boldsymbol{\lambda}\mathbf{L}}^T & 0 \\ A_{\mathbf{uL}} & A_{\mathbf{uu}} & B_{p_i\mathbf{u}}^T & A_{\boldsymbol{\lambda}\mathbf{u}}^T & 0 \\ 0 & B_{p_i\mathbf{u}} & 0 & B_{p_i\boldsymbol{\lambda}} & 0 \\ A_{\boldsymbol{\lambda}\mathbf{L}} & A_{\boldsymbol{\lambda}\mathbf{u}} & B_{p_i\boldsymbol{\lambda}}^T & A_{\boldsymbol{\lambda}\boldsymbol{\lambda}} & B_{p_{0e}\boldsymbol{\lambda}}^T \\ 0 & 0 & 0 & B_{p_{0e}\boldsymbol{\lambda}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{L} \\ \mathbf{u} \\ p_i \\ \boldsymbol{\lambda} \\ p_{0e} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ F_h \\ 0 \\ \mathbf{0} \\ 0 \end{bmatrix}. \tag{5.10}$$

Given the value of $\boldsymbol{\lambda}$ on $\partial\kappa$, $\mathbf{L}$, $\mathbf{u}$, p_i can be uniquely determined in each element κ. Namely, in the matrix form, we note that

$$\begin{bmatrix} A_{\mathbf{LL}} & A_{\mathbf{uL}}^T & 0 \\ A_{\mathbf{uL}} & A_{\mathbf{uu}} & B_{p_i\mathbf{u}}^T \\ 0 & B_{p_i\mathbf{u}} & 0 \end{bmatrix}$$

is block diagonal. Each block is nonsingular and corresponding to one element κ. Therefore, we can easily eliminate $\mathbf{L}$, $\mathbf{u}$ and p_i in each element independently from (5.10) and obtain the system for $\boldsymbol{\lambda}$ and p_{0e} only

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ p_{0e} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \tag{5.11}$$

where

$$A = A_{\boldsymbol{\lambda}\boldsymbol{\lambda}} - \begin{bmatrix} A_{\boldsymbol{\lambda}\mathbf{L}} & A_{\boldsymbol{\lambda}\mathbf{u}} & B_{p_i\boldsymbol{\lambda}}^T \end{bmatrix} \begin{bmatrix} A_{\mathbf{L}\mathbf{L}} & A_{\mathbf{u}\mathbf{L}}^T & 0 \\ A_{\mathbf{u}\mathbf{L}} & A_{\mathbf{u}\mathbf{u}} & B_{p_i\mathbf{u}}^T \\ 0 & B_{p_i\mathbf{u}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\boldsymbol{\lambda}\mathbf{L}}^T \\ A_{\boldsymbol{\lambda}\mathbf{u}}^T \\ B_{p_i\boldsymbol{\lambda}} \end{bmatrix},$$

and

$$B = B_{p_{0e}\boldsymbol{\lambda}}, \quad b = -\begin{bmatrix} A_{\boldsymbol{\lambda}\mathbf{L}} & A_{\boldsymbol{\lambda}\mathbf{u}} & B_{p_i\boldsymbol{\lambda}}^T \end{bmatrix} \begin{bmatrix} A_{\mathbf{L}\mathbf{L}} & A_{\mathbf{u}\mathbf{L}}^T & 0 \\ A_{\mathbf{u}\mathbf{L}} & A_{\mathbf{u}\mathbf{u}} & B_{p_i\mathbf{u}}^T \\ 0 & B_{p_i\mathbf{u}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ F_h \\ 0 \end{bmatrix}.$$

Once the solution $\boldsymbol{\lambda}$ and p_{0e} of (5.11) is obtained, the solution of (5.10) can be completed by computing $\mathbf{L}$, $\mathbf{u}$, p_i in each element with the given $\boldsymbol{\lambda}$.

By (Nguyen et al., 2010, Theorem 2.1), the system (5.11) can be considered as the matrix form of the following problem: to find $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ and $p_{0e} \in W_{0e}$ such that

$$\begin{cases} a_h(\boldsymbol{\lambda},\boldsymbol{\mu}) + b_h(p_{0e},\boldsymbol{\mu}) &= l_h(\boldsymbol{\mu}), & \forall\ \boldsymbol{\mu} \in \boldsymbol{\Lambda} \\ b_h(q_{0e},\boldsymbol{\lambda}) &= 0, & \forall\ q_{0e} \in W_{0e} \end{cases} \tag{5.12}$$

Here

$$\begin{aligned} a_h(\eta,\mu) &= \textstyle\sum_{\kappa\in\mathcal{T}_h} (\mathcal{L}(\eta),\mathcal{L}(\mu))_\kappa + \langle \tau_\kappa(\mathcal{U}(\eta)-\eta),(\mathcal{U}(\mu)-\mu)\rangle_{\partial\kappa}, \\ b_h(p_{0e},\mu) &= -\langle p_{0e}, \mu\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} \\ l_h(\mu) &= \textstyle\sum_{\kappa\in\mathcal{T}_h} l_\kappa(\eta,\mu) = \sum_{\kappa\in\mathcal{T}_h} (f_h,\mathcal{U}(\mu))_\kappa, \end{aligned} \tag{5.13}$$

where $\mathcal{L}(\mu)$ and $\mathcal{U}(\mu)$ are the unique solution ($\mathbf{L}_h = \mathcal{L}(\mu),\ \mathbf{u}_h = \mathcal{U}(\mu)$) of the local element problem (5.10) with $\boldsymbol{\lambda} = \mu$, $\mathbf{f} = \mathbf{0}$, and $\rho = 0$.

In next two sections, we will develop a BDDC algorithm to solve the system in (5.11) for $\boldsymbol{\lambda}$ and p_{0e}.

5.1.3 Reduced Subdomain Interface Problem

We decompose Ω into N nonoverlapping subdomain Ω_i with diameters H_i, $i = 1,\ldots,N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such elements forming an individual subdomain is uniformly bounded. We define edges/faces as open sets shared by two subdomains. Two nodes belong to the same edge/face when they are associated with the same pair of subdomains. Let Γ be the interface between the subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h := \left(\cup_{i\neq j}\partial\Omega_{i,h}\cap\partial\Omega_{j,h}\right)\setminus\partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is that of $\partial\Omega$. We assume the triangulation of each subdomain is quasi-uniform.

We decompose the velocity numerical trace $\boldsymbol{\Lambda}$ and the average pressure W_{0e} into:

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_I \oplus \widehat{\boldsymbol{\Lambda}}_\Gamma, \qquad W_{0e} = W_I \oplus W_0.$$

$\widehat{\boldsymbol{\Lambda}}_\Gamma$ denotes the degrees of freedom associated with Γ. $\boldsymbol{\Lambda}_I$ and W_I are products of subdomain interior velocity numerical trace spaces $V_I^{(i)}$ and subdomain interior pressure spaces $W_I^{(i)}$, respectively; i.e.,

$$\boldsymbol{\Lambda}_I = \prod_{i=1}^{N} \boldsymbol{\Lambda}_I^{(i)}, \qquad W_I = \prod_{i=1}^{N} W_I^{(i)}.$$

The elements of $\boldsymbol{\Lambda}_I^{(i)}$ are supported in the subdomain Ω_i and vanishes on its interface Γ_i, while the elements of $W_I^{(i)}$ are restrictions of the pressure variables to Ω_i which satisfy $\int_{\Omega_i} p_I^{(i)} = 0$. $\widehat{\boldsymbol{\Lambda}}_\Gamma$ is the subspace of edge/face functions on Γ in $\boldsymbol{\Lambda}$, and W_0 is the subspace of W with constant values $p_0^{(i)}$ in the subdomain Ω_i that satisfy $\sum_{i=1}^{N} p_0^{(i)} m(\Omega_i) = 0$, where $m(\Omega_i)$ is the measure of the subdomain Ω_i.

We denote the space of interface velocity numerical trace variables of the subdomain Ω_i by $\boldsymbol{\Lambda}_\Gamma^{(i)}$, and the associated product space by $\boldsymbol{\Lambda}_\Gamma = \prod_{i=1}^{N} \boldsymbol{\Lambda}_\Gamma^{(i)}$; generally edge/face functions in

$\boldsymbol{\Lambda}_\Gamma$ are discontinuous across the interface. We define the restriction operators $R_\Gamma^{(i)} : \widehat{\boldsymbol{\Lambda}}_\Gamma \to \boldsymbol{\Lambda}_\Gamma^{(i)}$ to be an operator which maps functions in the continuous global interface velocity numerical trace variable space $\widehat{\boldsymbol{\Lambda}}_\Gamma$ to the subdomain component space $\boldsymbol{\Lambda}_\Gamma^{(i)}$. Also, $R_\Gamma : \widehat{\boldsymbol{\Lambda}}_\Gamma \to \boldsymbol{\Lambda}_\Gamma$ is the direct sum of $R_\Gamma^{(i)}$.

The global problem (5.11) can be written as

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} & \widehat{B}_{0\Gamma}^T \\ 0 & 0 & \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_I \\ p_I \\ \boldsymbol{\lambda}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} b_I \\ 0 \\ b_\Gamma \\ 0 \end{bmatrix} \tag{5.14}$$

and it is assembled from subdomain problem

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Gamma I}^{(i)^T} & 0 \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\ A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} & B_{0\Gamma}^{(i)^T} \\ 0 & 0 & B_{0\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_I^{(i)} \\ p_I^{(i)} \\ \boldsymbol{\lambda}_\Gamma^{(i)} \\ p_0^{(i)} \end{bmatrix} = \begin{bmatrix} b_I^{(i)} \\ 0 \\ b_\Gamma^{(i)} \\ 0 \end{bmatrix}. \tag{5.15}$$

We can eliminate the subdomain interior variables $\boldsymbol{\lambda}_I^{(i)}$ and $p_I^{(i)}$ in each subdomain independently, and define the subdomain Schur complement $S_\Gamma^{(i)}$ as follows: given $\boldsymbol{\lambda}_\Gamma^{(i)} \in \boldsymbol{\Lambda}_\Gamma^{(i)}$, determine $S_\Gamma^{(i)}\boldsymbol{\lambda}_\Gamma^{(i)}$ such that

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)T} & A_{\Gamma I}^{(i)T} \\ B_{II}^{(i)} & 0 & B_{\Gamma I}^{(i)T} \\ A_{\Gamma I}^{(i)} & B_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_I^{(i)} \\ p_I^{(i)} \\ \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ S_\Gamma^{(i)}\boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix}. \tag{5.16}$$

The global interface problem is assembled from the subdomain interface problems, and can be written as: find $(\boldsymbol{\lambda}_\Gamma, p_0) \in \left(\widehat{\boldsymbol{\Lambda}}_\Gamma, W_0\right)$, such that

$$\widehat{S}\begin{bmatrix}\boldsymbol{\lambda}_\Gamma\\ p_0\end{bmatrix}=\begin{bmatrix}g_\Gamma\\ 0\end{bmatrix},\quad \text{where } \widehat{S}=\begin{bmatrix}\widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T\\ \widehat{B}_{0\Gamma} & 0\end{bmatrix}, \tag{5.17}$$

$\widehat{S}_\Gamma=\sum_{i=1}^N R_\Gamma^{(i)T}S_\Gamma^{(i)}R_\Gamma^{(i)}$, $\widehat{B}_{0\Gamma}=\sum_{i=1}^N B_{0\Gamma}^{(i)}R_\Gamma^{(i)}$, and

$$g_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)T}\left\{b_\Gamma^{(i)}-\begin{bmatrix}A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)T}\end{bmatrix}\begin{bmatrix}A_{II}^{(i)} & B_{II}^{(i)T}\\ B_{II}^{(i)} & 0\end{bmatrix}^{-1}\begin{bmatrix}b_I^{(i)}\\ 0\end{bmatrix}\right\}.$$

The operator $\widehat{S}_\Gamma$ is symmetric positive definite, because of the Dirichlet boundary conditions on $\partial\Omega$ and the primal continuity constraints defined on the interface. Note that $\widehat{S}$ is symmetric indefinite. In what follows, we will propose a BDDC preconditioner, and show that the preconditioned operator is positive definite when restricted to a proper subspace. A preconditioned conjugate gradient method can then be used to solve the global interface problem.

5.1.4 The BDDC Preconditioner

The BDDC (Balancing Domain Decomposition by Constraints) methods, which were introduced and analyzed by Dohrmann, Mandel, and Tezaur in (Dohrmann, 2003; Mandel & Dohrmann, 2003; Mandel et al., 2005), are originally designed for standard finite element discretization of elliptic problems. The BDDC algorithms are similar to the balancing Neumann-Neumann algorithms. However, their coarse problems, in BDDC, are given in terms of sets of primal constraints. The main advantage of such coarse problem is that the local subdomain problems, arising in the BDDC algorithms, are invertible. They are one of the most tested and popular domain decomposition algorithms and suitable for parallel computation.

In order to introduce the BDDC preconditioner, we first introduce a partially assembled

interface space $\widetilde{\mathbf{\Lambda}}_\Gamma$ by

$$\widetilde{\mathbf{\Lambda}}_\Gamma = \widehat{\mathbf{\Lambda}}_\Pi \oplus \mathbf{\Lambda}_\Delta = \widehat{\mathbf{\Lambda}}_\Pi \oplus \prod_{i=1}^{N} \mathbf{\Lambda}_\Delta^{(i)}.$$

Here, $\widehat{\mathbf{\Lambda}}_\Pi$ is the coarse level, primal interface velocity space and the space $\mathbf{\Lambda}_\triangle$ is the direct sum of the $\mathbf{\Lambda}_\triangle^{(i)}$, which are spanned by the remaining interface degrees of freedom.In the space $\widetilde{\mathbf{\Lambda}}_\Gamma$, we relax most continuity constraints across the interface but retain the continuity at the primal unknowns, which makes all the linear systems nonsigular.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)} : \widetilde{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\mathbf{\Lambda}}_\Gamma$ to the components $\mathbf{\Lambda}_\Gamma^{(i)}$ of the subdomain Ω_i. $\overline{R}_\Gamma : \widetilde{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Gamma$ is the direct sum of $\overline{R}_\Gamma^{(i)}$. $R_\Delta^{(i)} : \widehat{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Delta^{(i)}$ maps the functions from $\widehat{\mathbf{\Lambda}}_\Gamma$ to $\mathbf{\Lambda}_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi} : \widehat{\mathbf{\Lambda}}_\Gamma \to \widehat{\mathbf{\Lambda}}_\Pi$ is a restriction operator from $\widehat{\mathbf{\Lambda}}_\Gamma$ to its subspace $\widehat{\mathbf{\Lambda}}_\Pi$. $\widetilde{R}_\Gamma : \widehat{\mathbf{\Lambda}}_\Gamma \to \widetilde{\mathbf{\Lambda}}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define the positive scaling factor $\delta_i^\dagger(x)$ as follows:

$$\delta_i^\dagger(x) = \frac{1}{card\,(\mathcal{I}_x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{I}_x$ is the set of indices of the subdomains that have x on their boundaries, and $card\,(\mathcal{I}_x)$ counts the number of the subdomain boundaries to which x belongs. We note that $\delta_i^\dagger(x)$ is constant on each edge. Multiplying each row of $R_\Delta^{(i)}$ with the scaling factor gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{D,\Delta}^{(i)}$.

We denote the direct sum of $S_\Gamma^{(i)}$ by S_Γ and the partially assembled interface velocity Schur complement is defined by $\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma$. Correspondingly, we define an operator $\widetilde{B}_{0\Gamma}$, which maps the partially assembled interface velocity space $\widetilde{\mathbf{\Lambda}}_\Gamma$ into the space of right-hand sides corresponding to W_0. $\widetilde{B}_{0\Gamma}$ is obtained from the subdomain operators $B_{0\Gamma}^{(i)}$ by assembling with respect to the primal interface velocity part, i.e., $\widetilde{B}_{0\Gamma} = \sum_{i=1}^{N} B_{0\Gamma}^{(i)} \overline{R}_\Gamma^{(i)}$. Using the following notation

$$\widetilde{R}_D = \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix}, \qquad \widetilde{S} = \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix}, \tag{5.18}$$

the preconditioner for solving the global interface Stokes problem is

$$M^{-1} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D.$$

and the preconditioned BDDC algorithm is then of the form: find $(\boldsymbol{\lambda}_\Gamma, p_0) \in \left(\widehat{\boldsymbol{\Lambda}}_\Gamma, W_0\right)$, such that

$$\widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \widehat{S} \begin{bmatrix} \boldsymbol{\lambda}_\Gamma \\ p_0 \end{bmatrix} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \begin{bmatrix} g_\Gamma \\ 0 \end{bmatrix}. \tag{5.19}$$

Note that $\widetilde{R}_{D,\Gamma}$ is of full rank and that the preconditioner is nonsingular.

The matrix $\widehat{S}$ defined in (5.17) is symmetric indefinite on the space $\left(\widehat{\boldsymbol{\Lambda}}_\Gamma, W_0\right)$, but it is positive semi-definite on the subspace $\left(\widehat{\boldsymbol{\Lambda}}_\Gamma \cap \mathbf{ker}\,(\widehat{B}_{0\Gamma}), W_0\right)$. With the careful chosen primal velocity space $\widehat{\boldsymbol{\Lambda}}_\Pi$, we can construct a BDDC preconditioner to make sure the preconditioned BDDC operator in (5.19) is symmetric positive definite in a subspace and the conjugate gradient iterates remain in this subspace when solving (5.19). This subspace is called the benign subspace.

Definition 5.1.1 (Benign Subspaces)**.** *We will call*

$$\widehat{\boldsymbol{\Lambda}}_{\Gamma,B} = \{\boldsymbol{\lambda}_\Gamma \in \widehat{\boldsymbol{\Lambda}}_\Gamma \mid \widehat{B}_{0\Gamma}\boldsymbol{\lambda}_\Gamma = 0\}, \quad \widetilde{\boldsymbol{\Lambda}}_{\Gamma,B} \ = \{\boldsymbol{\lambda}_\Gamma \in \widetilde{\boldsymbol{\Lambda}}_\Gamma \mid \widetilde{B}_{0\Gamma}\boldsymbol{\lambda}_\Gamma = 0\}$$

the benign subspaces of $\widehat{\boldsymbol{\Lambda}}_\Gamma$ *and* $\widetilde{\boldsymbol{\Lambda}}_\Gamma$*, respectively.*

It is easy to see that the operators $\widehat{S}$ and $\widetilde{S}$, defined in (5.17) and (5.19), are symmetric positive definite on $\left(\widehat{\boldsymbol{\Lambda}}_{\Gamma,B}, W_0\right)$ and $\left(\widetilde{\boldsymbol{\Lambda}}_{\Gamma,B}, W_0\right)$, respectively.

As in (Li & Widlund, 2006a; Tu, 2005), in order to make the iterates in the benign subspace, we require that the functions in the dual velocity space satisfies the *no-net flux*

condition, i.e.,

$$\int_{\partial\Omega_i} \boldsymbol{\lambda}_{\Delta}^{(i)} \cdot \boldsymbol{n} = 0, \ \forall \ \boldsymbol{\lambda}_{\Delta}^{(i)} \in \boldsymbol{\Lambda}_{\Delta}^{(i)}. \tag{5.20}$$

In order to make the dual velocity space satisfy the *no-net flux condition*, we choose the primal variables which are spanned by subdomain interface edge/face basis functions with constant values at the nodes of the edge/face for two/three dimensions. We change the variables so that the degrees of freedom of each primal constraint is explicit; see (Li & Widlund, 2006b; Klawonn & Widlund, 2006). The dual space $\boldsymbol{\Lambda}_{\triangle}$ are spanned by the remaining interface degrees of freedom, with a zero average over each edge/face.

The following Lemma, see (Li & Widlund, 2006a, Lemma 6.2) and (Tu, 2005, Lemma 4.1), is crucial to prove the positive definiteness of the preconditioned BDDC operator.

Lemma 5.1.1. *Let* $\boldsymbol{\lambda}_{\Gamma} \in \widetilde{\boldsymbol{\Lambda}}_{\Gamma,B}$. *Then,* $\widetilde{R}_{D,\Gamma}^{T}\boldsymbol{\lambda}_{\Gamma} \in \widehat{\boldsymbol{\Lambda}}_{\Gamma,B}$.

5.1.5 Some Auxiliary Results

This section we collect a number of results which are needed in our condition number estimate of the preconditioned system (5.19). We define

$$\gamma_{h,\tau} = \max_{\kappa \in \mathcal{T}_h} \{1 + \tau_\kappa h_\kappa\}, \tag{5.21}$$

where τ_κ and h_κ are the stabilization parameter and the diameter of the element κ, respectively. We use c and C to present constants which are independent of h, H, and τ_κ.

Let $\mathbf{Q}_h$ and $\mathbf{Q}_0$ be the L^2 projection operators from $\left[L^2(\kappa)\right]^n$ onto $\mathcal{P}_k(\kappa)$ and $\mathbf{P}_k(\kappa)$, respectively. $\mathbf{Q}_b$ is the L^2 projection from $\left[L^2(e)\right]^n$ onto $\mathbf{P}_k(e)$. Let $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity assumption as specified in (Wang & Ye, 2014, lemma 4.1). The following lemma is (Wang & Ye, 2014, Lemma 4.1) or (Wang & Ye, 2016, Lemma A.1).

Lemma 5.1.2. $\mathbf{w} \in \left[H^{r+1}(\Omega)\right]^n$ *with* $1 \leq r \leq k$. *Then, for* $0 \leq s \leq 1$, *we have*

$$\sum_{\kappa \in \mathcal{T}_h} h_\kappa^{2s} \|\mathbf{w} - \mathbf{Q}_0 \mathbf{w}\|^2_{H^s(\kappa)} \leq h^{2(r+1)} \|\mathbf{w}\|^2_{H^{r+1}(\Omega)}. \tag{5.22}$$

Let κ be an element with e as an edge/face. For any function $g \in H^1(\kappa)$, the following trace inequality holds (Wang & Ye, 2014, Lemma A.3) or (Wang & Ye, 2016, Equation (A.4)).

Lemma 5.1.3.

$$\|g\|^2_{L^2(e)} \leq C \left(h_\kappa^{-1} \|g\|^2_{L^2(\kappa)} + h_\kappa \|\nabla g\|^2_{L^2(\kappa)} \right). \tag{5.23}$$

We first introduce several useful norms and semi-norms, which are defined in (Li & Widlund, 2006a; Tu, 2005).

The subdomain Schur complements $S_\Gamma^{(i)}$, defined in (5.16), are symmetric, positive semi-definite by the inertia of Schur complements. They are singular for any subdomains with a boundary that does not intersect $\partial\Omega$.

The operators $\widehat{S}_\Gamma$ and $\widetilde{S}_\Gamma$, defined in (5.17) and (5.18), are symmetric positive definite because of the Dirichlet boundary condition on $\partial\Omega$ and sufficiently many primal continuity constraints for the *no-net-flux condition*.

The interface operators $\widehat{S}$ and $\widetilde{S}$, defined in (5.17) and (5.18), are indefinite, but they are positive, semi-definite when they are restricted to the benign subspaces of $\left(\widehat{\mathbf{\Lambda}}, W_0\right)$ and $\left(\widetilde{\mathbf{\Lambda}}, W_0\right)$, respectively. We can define

$$\begin{aligned} |\mathbf{w}|^2_{\widehat{S}} &= \mathbf{w}^T \widehat{S} \mathbf{w} = \|\boldsymbol{\lambda}_\Gamma\|^2_{\widehat{S}_\Gamma}, \qquad && \forall\ \mathbf{w} = (\boldsymbol{\lambda}_\Gamma, p_0) \in \left(\widehat{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right), \\ |\mathbf{w}|^2_{\widetilde{S}} &= \mathbf{w}^T \widetilde{S} \mathbf{w} = \|\boldsymbol{\lambda}_\Gamma\|^2_{\widetilde{S}_\Gamma}, \qquad && \forall\ \mathbf{w} = (\boldsymbol{\lambda}_\Gamma, p_0) \in \left(\widetilde{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right). \end{aligned} \tag{5.24}$$

We also define $S^{(i)}_{\Gamma,E}$, the subdomain Schur complement for the corresponding elliptic

problem, as follows: given $\boldsymbol{\lambda}_\Gamma^{(i)} \in \boldsymbol{\Lambda}_\Gamma^{(i)}$, determine $S_{\Gamma,E}^{(i)}\boldsymbol{\lambda}_\Gamma^{(i)}$ such that

$$A_a^{(i)} \begin{bmatrix} \mathbf{L}^{(i)} \\ \mathbf{u}^{(i)} \\ \boldsymbol{\lambda}_I^{(i)} \\ \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ S_{\Gamma,E}^{(i)}\boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix}, \tag{5.25}$$

where $A_a^{(i)}$ is the subdomain matrix for A_a defined in (5.8) and defined as follows:

$$A_a^{(i)} = \begin{bmatrix} A_{\mathbf{LL}}^{(i)} & A_{\mathbf{uL}}^{(i)^T} & A_{I\mathbf{L}}^{(i)^T} & A_{\Gamma\mathbf{L}}^{(i)T} \\ A_{\mathbf{uL}}^{(i)} & A_{\mathbf{uu}}^{(i)} & A_{I\mathbf{u}}^{(i)^T} & A_{\Gamma\mathbf{u}}^{(i)T} \\ A_{I\mathbf{L}}^{(i)} & A_{I\mathbf{u}}^{(i)} & A_{II}^{(i)} & A_{\Gamma I}^{(i)T} \\ A_{\Gamma\mathbf{L}}^{(i)} & A_{\Gamma\mathbf{u}}^{(i)} & A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}. \tag{5.26}$$

Let $\boldsymbol{\lambda}^{(i)} = \begin{bmatrix} \boldsymbol{\lambda}_I^{(i)} & \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix}^T$, by the definition in (5.6), (Cockburn et al., 2009a, Theorem 2.1) or (Cockburn et al., 2014, Theorem 2.1), we have

$$\begin{aligned} \left\| \begin{bmatrix} \mathbf{L}^{(i)} \\ \mathbf{u}^{(i)} \\ \boldsymbol{\lambda}_I^{(i)} \\ \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix} \right\|_{A_a^{(i)}}^2 &= \begin{bmatrix} \mathbf{L}^{(i)} \\ \mathbf{u}^{(i)} \\ \boldsymbol{\lambda}_I^{(i)} \\ \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix}^T A_a^{(i)} \begin{bmatrix} \mathbf{L}^{(i)} \\ \mathbf{u}^{(i)} \\ \boldsymbol{\lambda}_I^{(i)} \\ \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix} = \boldsymbol{\lambda}_\Gamma^{(i)^T} S_{\Gamma,E}^{(i)} \lambda_\Gamma^{(i)} = \\ &= \sum_{\kappa \in \mathcal{T}_h(\Omega_i)} (\mathbf{L}^{(i)}, \mathbf{L}^{(i)})_\kappa + \tau_\kappa \left\langle \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)}, \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)} \right\rangle_{\partial\kappa}, \end{aligned} \tag{5.27}$$

for $\begin{bmatrix} \mathbf{L}^{(i)} & \mathbf{u}^{(i)} & \boldsymbol{\lambda}_I^{(i)} & \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix}^T$ satisfying (5.25).

We also have the following lemma:

Lemma 5.1.4. *If $1 \leq Ch_\kappa\tau_\kappa$, for all $\kappa \in \mathcal{T}_h(\Omega_i)$, then*

$$\sum_{\kappa\in\mathcal{T}_h(\Omega_i)} \|\nabla \mathbf{u}^{(i)}\|^2_{L^2(\kappa)} \leq C \left\| \begin{bmatrix} \mathbf{L}^{(i)} \\ \mathbf{u}^{(i)} \\ \boldsymbol{\lambda}_I^{(i)} \\ \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix} \right\|^2_{A_a^{(i)}}, \tag{5.28}$$

for $\begin{bmatrix} \mathbf{L}^{(i)} & \mathbf{u}^{(i)} & \boldsymbol{\lambda}_I^{(i)} & \boldsymbol{\lambda}_\Gamma^{(i)} \end{bmatrix}^T$ *satisfying (5.25).*

Proof:

$$\begin{aligned}
& (\nabla \mathbf{u}^{(i)}, \nabla \mathbf{u}^{(i)})_\kappa = -(\mathbf{u}^{(i)}, \nabla\cdot\nabla \mathbf{u}^{(i)})_\kappa + \left\langle \mathbf{u}^{(i)}, \nabla \mathbf{u}^{(i)} \cdot \boldsymbol{n} \right\rangle_{\partial\kappa} \\
= \;& -(\mathbf{u}^{(i)}, \nabla\cdot\nabla \mathbf{u}^{(i)})_\kappa + \left\langle \boldsymbol{\lambda}^{(i)}, \nabla \mathbf{u}^{(i)} \cdot \boldsymbol{n} \right\rangle_{\partial\kappa} + \left\langle \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)}, \nabla \mathbf{u}^{(i)} \cdot \boldsymbol{n} \right\rangle_{\partial\kappa} \\
= \;& (\mathbf{L}^{(i)}, \nabla \mathbf{u}^{(i)})_\kappa + \left\langle \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)}, \nabla \mathbf{u}^{(i)} \cdot \boldsymbol{n} \right\rangle_{\partial\kappa} \\
\leq \;& (\mathbf{L}^{(i)}, \mathbf{L}^{(i)})^{1/2}_\kappa (\nabla \mathbf{u}^{(i)}, \nabla \mathbf{u}^{(i)})^{1/2}_\kappa + h_\kappa^{-1/2} \left\langle \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)}, \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)} \right\rangle^{1/2}_{\partial\kappa} h_\kappa^{1/2} \left\langle \nabla \mathbf{u}^{(i)} \cdot \boldsymbol{n}, \nabla \mathbf{u}^{(i)} \cdot \boldsymbol{n} \right\rangle^{1/2}_{\partial\kappa} \\
\leq \;& (\mathbf{L}^{(i)}, \mathbf{L}^{(i)})^{1/2}_\kappa (\nabla \mathbf{u}^{(i)}, \nabla \mathbf{u}^{(i)})^{1/2}_\kappa + Ch_\kappa^{-1/2} \left\langle \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)}, \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)} \right\rangle^{1/2}_{\partial\kappa} \left(\nabla \mathbf{u}^{(i)}, \nabla \mathbf{u}^{(i)} \right)^{1/2}_\kappa,
\end{aligned}$$

where we use integration by part for the first equality, the first equation in (5.5) for the third equality, the trace inequality (Lemma 5.1.3) and the inverse inequality for the last inequality.

Therefore, if $1 \leq Ch_\kappa\tau_\kappa$,

$$(\nabla \mathbf{u}^{(i)}, \nabla \mathbf{u}^{(i)})_\kappa \;\leq\; C\left((\mathbf{L}^{(i)}, \mathbf{L}^{(i)})_\kappa + \tau_\kappa \left\langle \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)}, \mathbf{u}^{(i)} - \boldsymbol{\lambda}^{(i)} \right\rangle_{\partial\kappa} \right).$$

□

Similar to the inf-sup condition of the weak Galerkin finite element methods (Wang & Ye, 2016, Lemma 4.3), we have the following theorem:

Theorem 5.1.5. *If $h_\kappa \tau_\kappa \leq C$, then there exists a positive constant β independent of h and H, such that*

$$\sup_{u_a \in (\mathbf{G},\mathbf{V},\mathbf{\Lambda})} \frac{u_a^T B_a^T p}{(u_a^T A_a u_a)^{1/2}} \geq \beta \|p\|_{L^2(\Omega)}, \tag{5.29}$$

for all $p \in W$. Here A_a, B_a are defined in (5.8). The theorem is also hold when Ω is replaced by the subdomain Ω_i.

Proof: For any given $p \in W \subset L_0^2(\Omega)$, it is well known that there exists a vector-valued function $\tilde{v} \in [H_0^1(\Omega)]^n$ such that

$$\frac{(\nabla \cdot \tilde{v}, p)}{\|\tilde{v}\|_{H^1(\Omega)}} \geq C \|p\|_{L^2(\Omega)}, \tag{5.30}$$

where C is a constant depending only on the domain Ω.

Let $\mathbf{L} = \mathbf{Q}_h \nabla \tilde{v}$, $\mathbf{v} = \mathbf{Q}_0 \tilde{v}$, $\boldsymbol{\lambda} = \mathbf{Q}_b \tilde{v}$, and $u_a^T = \begin{bmatrix} \mathbf{L} & \mathbf{v} & \boldsymbol{\lambda} \end{bmatrix}$. Recall $\mathbf{Q}_h$, $\mathbf{Q}_0$, $\mathbf{Q}_b$ are the L_2 projections to the corresponding spaces. For any $\mathbf{G}_h \in \mathbf{G}$, we have

$$\begin{aligned} & (\mathbf{L}, \mathbf{G}_h)_\kappa + (\mathbf{v}, \nabla \cdot \mathbf{G}_h) - \langle \boldsymbol{\lambda}, \mathbf{G}_h \mathbf{n} \rangle \qquad (5.31) \\ = \; & (\mathbf{Q}_h \nabla \tilde{v}, \mathbf{G}_h)_\kappa + (\mathbf{Q}_0 \tilde{v}, \nabla \cdot \mathbf{G}_h) - \langle \mathbf{Q}_b \tilde{v}, \mathbf{G}_h \mathbf{n} \rangle \\ = \; & (\nabla \tilde{v}, \mathbf{G}_h)_\kappa + (\tilde{v}, \nabla \cdot \mathbf{G}_h) - \langle \tilde{v}, \mathbf{G}_h \mathbf{n} \rangle \\ = \; & 0. \end{aligned}$$

Therefore, if $\tau_\kappa h_\kappa \leq C$, we have

$$
\begin{aligned}
u_a^T A_a u_a &= \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\mathbf{L},\mathbf{L})_\kappa + \tau_\kappa (\langle \mathbf{v},\mathbf{v}\rangle_{\partial\kappa} - 2\langle \boldsymbol{\lambda},\mathbf{v}\rangle_{\partial\kappa} + \langle \boldsymbol{\lambda},\boldsymbol{\lambda}\rangle_{\partial\kappa})) \\
&= \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\mathbf{L},\mathbf{L})_\kappa + \tau_\kappa \langle \mathbf{v}-\boldsymbol{\lambda},\mathbf{v}-\boldsymbol{\lambda}\rangle_{\partial\kappa}) \\
&= \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\mathbf{Q}_h\nabla\tilde{v},\mathbf{Q}_h\nabla\tilde{v})_\kappa + \tau_\kappa \langle \mathbf{Q}_0\tilde{v}-\mathbf{Q}_b\tilde{v},\mathbf{Q}_0\tilde{v}-\mathbf{Q}_b\tilde{v}\rangle_{\partial\kappa}) \\
&\leq \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\nabla\tilde{v},\nabla\tilde{v})_\kappa + \tau_\kappa \langle \mathbf{Q}_b(\mathbf{Q}_0\tilde{v}-\tilde{v}),\mathbf{Q}_b(\mathbf{Q}_0\tilde{v}-\tilde{v})\rangle_{\partial\kappa}) \\
&\leq \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\nabla\tilde{v},\nabla\tilde{v})_\kappa + \tau_\kappa \langle \mathbf{Q}_0\tilde{v}-\tilde{v},\mathbf{Q}_0\tilde{v}-\tilde{v}\rangle_{\partial\kappa}) \\
&\leq C \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\nabla\tilde{v},\nabla\tilde{v})_\kappa + h_\kappa^{-1} \|\mathbf{Q}_0\tilde{v}-\tilde{v}\|^2_{L^2(\partial\kappa)}) \\
&\leq C \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\nabla\tilde{v},\nabla\tilde{v})_\kappa + h_\kappa^{-2} \|\mathbf{Q}_0\tilde{v}-\tilde{v}\|^2_{L^2(\kappa)} + \|\nabla(\mathbf{Q}_0\tilde{v}-\tilde{v})\|^2_{L^2(\kappa)}) \\
&\leq C\|\tilde{v}\|^2_{H^1(\Omega)}.
\end{aligned}
\tag{5.32}
$$

Here, we use (5.31) for the first equality, the definitions of $\mathbf{L}$, $\mathbf{v}$, and $\boldsymbol{\lambda}$ for the third equality, the definitions of the projections for fourth and fifth inequalities, the choice of τ_κ for the sixth inequality, the trace inequality (5.23) for the seventh inequality, (5.22) for the last inequality.

$$
\begin{aligned}
u_a^T B_a^T p &= (B_a u_a)^T p = \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\mathbf{v},\nabla p)_\kappa - \langle \boldsymbol{\lambda}\cdot\boldsymbol{n},p\rangle_{\partial\kappa}) \\
&= \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\mathbf{Q}_0\tilde{v},\nabla p)_\kappa - \langle \mathbf{Q}_b\tilde{v}\cdot\boldsymbol{n},p\rangle_{\partial\kappa}) \\
&= \sum_{\kappa\in\mathcal{T}_h(\Omega)} ((\tilde{v},\nabla p)_\kappa - \langle \tilde{v}\cdot\boldsymbol{n},p\rangle_{\partial\kappa}) \\
&= \sum_{\kappa\in\mathcal{T}_h(\Omega)} -(\nabla\cdot\tilde{v},p)_\kappa = -(\nabla\cdot\tilde{v},p).
\end{aligned}
\tag{5.33}
$$

Combining (5.32) and (5.33), we have

$$\frac{|u_a B_a^T p|}{(u_a^T A_q u_a)^{1/2}} \geq \frac{(\nabla \cdot \tilde{v}, p)}{\|\tilde{v}\|_{H^1(\Omega)}} \geq \beta \|p\|_{L^2(\Omega)}. \tag{5.34}$$

□

Let $\left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}} = \boldsymbol{\lambda}_\Gamma^{(i)T} S_\Gamma^{(i)} \boldsymbol{\lambda}_\Gamma^{(i)}$, and $\left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_{\Gamma,E}^{(i)}} = \boldsymbol{\lambda}_\Gamma^{(i)T} S_{\Gamma,E}^{(i)} \boldsymbol{\lambda}_\Gamma^{(i)}$. We will use the inf-sup stability for each subdomain in the proof of the following lemma. Similar results for the standard finite element discretization can be found in (Bramble & Pasciak, 1989).

Lemma 5.1.6. *If $\tau_\kappa h_\kappa \approx C$, the*

$$c\frac{\beta^2}{(1+\beta)^2}\left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}} \leq \left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_{\Gamma,E}^{(i)}} \leq \left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}}, \qquad \forall \boldsymbol{\lambda}_\Gamma^{(i)} \in \boldsymbol{\Lambda}_\Gamma^{(i)},$$

where β is the inf-sup stability constant defined in (5.29).

Proof. We define the harmonic and Stokes extensions from $\boldsymbol{\Lambda}_\Gamma^{(i)}$ to $(\mathbf{G}^{(i)}, \mathbf{V}^{(i)}, \boldsymbol{\Lambda}^{(i)})$ as follows: for any $\boldsymbol{\lambda}_\Gamma^{(i)} \in \boldsymbol{\Lambda}_\Gamma^{(i)}$,

$$\left|\mathcal{H}\boldsymbol{\lambda}_\Gamma^{(i)}\right|_{A_a^{(i)}} = \inf_{\mathbf{E}^{(i)}=(\mathbf{L}^{(i)},\mathbf{v}^{(i)},\boldsymbol{\lambda}^{(i)})\in(\mathbf{G}^{(i)},\mathbf{V}^{(i)},\boldsymbol{\Lambda}^{(i)}),\ \mathbf{E}^{(i)}|_{\Gamma_i}=\boldsymbol{\lambda}_\Gamma^{(i)}} \left|\mathbf{E}^{(i)}\right|_{A_a^{(i)}}, \tag{5.35}$$

and

$$\left|\mathcal{S}\boldsymbol{\lambda}_\Gamma^{(i)}\right|_{A_a^{(i)}} = \inf_{\mathbf{E}^{(i)}=(\mathbf{L}^{(i)},\mathbf{v}^{(i)},\boldsymbol{\lambda}^{(i)})\in(\mathbf{G}^{(i)},\mathbf{V}^{(i)},\boldsymbol{\Lambda}^{(i)}),\ \mathbf{E}^{(i)}|_{\Gamma_i}=\boldsymbol{\lambda}_\Gamma^{(i)}, B_{a,c}^{(i)}\mathbf{E}^{(i)}=0} \left|\mathbf{E}^{(i)}\right|_{A_a^{(i)}}, \tag{5.36}$$

where $A_a^{(i)}$, defined in (5.26), is the subdomain local matrix of A_a defined in (5.8). $B_{a,c}^{(i)}$ is the subdomain local matrix of B_a defined in (5.8) excluding the pressure degree of freedom corresponding to the subdomain average pressure. Given $\boldsymbol{\lambda}_\Gamma^{(i)}$, the harmonic extension $\mathcal{H}\boldsymbol{\lambda}_\Gamma^{(i)}$ can be obtained by solving the equations corresponding to the first three rows of (5.25). The Stokes extension $\mathcal{S}\boldsymbol{\lambda}_\Gamma^{(i)}$ can be obtained by first solving the equations corresponding to the first two rows of (5.16) and then using the resulting $[\boldsymbol{\lambda}_I^{(i)}\ \boldsymbol{\lambda}_\Gamma^{(i)}]^T$ to obtain $[\mathbf{L}^{(i)}\ \mathbf{v}^{(i)}\ p_i^{(i)}]^T$ in

each element. We denote the resulting pressure by p^* and let $p = \begin{bmatrix} p^* \\ 0 \end{bmatrix}$, which is the whole degrees of freedom for pressure in the subdomain Ω_i.

By the definition of these two extensions and the semi-norms $|\cdot|_{S_{\Gamma,E}}$ and $|\cdot|_{S_\Gamma}$, we have

$$\left|\mathcal{H}\boldsymbol{\lambda}_\Gamma^{(i)}\right|_{A_a^{(i)}} = \left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_{\Gamma,E}^{(i)}} \quad \text{and} \quad \left|\mathcal{S}\boldsymbol{\lambda}_\Gamma^{(i)}\right|_{A_a^{(i)}} = \left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}}.$$

The second inequality can be obtained as

$$\left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_{\Gamma,E}^{(i)}} = \left|\mathcal{H}\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{A_a^{(i)}} \leq \left|\mathcal{S}\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{A_a^{(i)}} = \left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}}.$$

We prove the first inequality as follows. Let $(\mathbf{L}_h^{(i)}, \mathbf{v}_h^{(i)}, \boldsymbol{\lambda}_h^{(i)}) = \mathbf{E}_h^{(i)} = \mathcal{H}\boldsymbol{\lambda}_\Gamma^{(i)}$, $(\mathbf{L}_s^{(i)}, \mathbf{v}_s^{(i)}, \boldsymbol{\lambda}_s^{(i)}) = \mathbf{E}_s^{(i)} = \mathcal{S}\boldsymbol{\lambda}_\Gamma^{(i)}$. By the definitions of the harmonic and Stokes extensions, we have

$$\begin{bmatrix} \mathbf{E}_s^{(i)} - \mathbf{E}_h^{(i)} \\ 0 \end{bmatrix}^T \begin{bmatrix} A_a^{(i)} & {B_a^{(i)}}^T \\ B_a^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_s^{(i)} \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, we have

$$|\mathbf{E}_s|^2_{A_a^{(i)}} = \mathbf{E}_h^T A_a^{(i)} \mathbf{E}_s + \mathbf{E}_h^T {B_a^{(i)}}^T p - \left(B_a^{(i)} \mathbf{E}_s\right)^T p. \tag{5.37}$$

Since $p = \begin{bmatrix} p^* \\ 0 \end{bmatrix}$, $\left(B_a^{(i)}\mathbf{E}_s\right)^T p = 0$, and,

$$\begin{aligned}
\left(B_a^{(i)}\mathbf{E}_h\right)^T p &= \sum_{\kappa\in\mathcal{T}_h(\Omega_i)} ((\mathbf{v}_h, \nabla p)_\kappa - \langle \boldsymbol{\lambda}_h \cdot \boldsymbol{n}, p\rangle_{\partial\kappa}) \\
&= \sum_{\kappa\in\mathcal{T}_h(\Omega_i)} -((\nabla\cdot\mathbf{v}_h, p)_\kappa + \langle (\boldsymbol{\lambda}_h - \mathbf{v}_h)\cdot\boldsymbol{n}, p\rangle_{\partial\kappa}) \\
&\leq C\left(\sum_{\kappa\in\mathcal{T}_h(\Omega_i)} (\nabla\mathbf{v}_h, \nabla\mathbf{v}_h)_\kappa\right)^{1/2}\left(\sum_{\kappa\in\mathcal{T}_h} (p,p)_\kappa\right)^{1/2} \\
&\quad + C\left(\sum_{\kappa\in\mathcal{T}_h(\Omega_i)} h_\kappa^{-1}\langle \boldsymbol{\lambda}_h - \mathbf{v}_h, \boldsymbol{\lambda}_h - \mathbf{v}_h\rangle_{\partial\kappa}\right)^{1/2}\left(\sum_{\kappa\in\mathcal{T}_h(\Omega_i)} h_\kappa \langle p,p\rangle_{\partial\kappa}\right)^{1/2} \\
&\leq C|\mathbf{E}_h|_{A_a^{(i)}}\|p\|_{L^2(\Omega_i)}, \qquad (5.38)
\end{aligned}$$

where we use the Cauchy-Schwarz inequality for the third inequality, (5.27), Lemmas 5.1.4 and 5.1.3 for the last inequality.

By (5.37), the Cauchy-Schwarz inequality, and (5.38), we have

$$|\mathbf{E}_s|^2_{A_a^{(i)}} = \mathbf{E}_h^T A_a^{(i)}\mathbf{E}_s + \mathbf{E}_h^T {B_a^{(i)}}^T p \leq C\left(|\mathbf{E}_h|_{A_a^{(i)}}|\mathbf{E}_s|_{A_a^{(i)}} + |\mathbf{E}_h|_{A_a^{(i)}}\|p\|_{L^2(\Omega_i)}\right). \qquad (5.39)$$

Applying Theorem 5.1.5 on the subdomain Ω_i, we have

$$\begin{aligned}
\| p \|_{L^2(\Omega_i)} &\leq \beta^{-1}\sup_{u_a\in(\mathbf{G}^{(i)},\mathcal{V}^{(i)},\boldsymbol{\Lambda}^{(i)})} \frac{u_a^T {B_a^{(i)}}^T p}{\left(u_a^T A_a^{(i)} u_a\right)^{1/2}} \\
&= \beta^{-1}\sup_{u_a\in(\mathbf{G}^{(i)},\mathcal{V}^{(i)},\boldsymbol{\Lambda}^{(i)})} \frac{u_a^T A_a^{(i)}\mathbf{E}_s}{\left(u_a^T A_a^{(i)} u_a\right)^{1/2}} \qquad (5.40) \\
&\leq \beta^{-1}|\mathbf{E}_s|_{A_a^{(i)}},
\end{aligned}$$

where we use the definition of the Stokes extension $\mathbf{E}_s$ for the second equality and the Cauchy-Schwarz inequality for the last inequality.

Plugging (5.40) into (5.39), we obtain

$$|\mathbf{E}_s|^2_{A_a^{(i)}} \;\leq\; C\left(|\mathbf{E}_h|_{A_a^{(i)}}\left|\mathbf{E}_s\right|_{A_a^{(i)}} + |\mathbf{E}_h|_{A_a^{(i)}}\beta^{-1}|\mathbf{E}_s|_{A_a^{(i)}}\right).$$

Therefore, we have

$$\frac{\beta^2}{(1+\beta)^2}\left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}} = \frac{\beta^2}{(1+\beta)^2}|\mathbf{E}_s|^2_{A_a^{(i)}} \leq C\left|\mathbf{E}_h\right|^2_{A_a^{(i)}} = C\left|\boldsymbol{\lambda}_\Gamma^{(i)}\right|^2_{S_{\Gamma,E}^{(i)}}. \tag{5.41}$$

□

In order to prove the condition number bounds for the BDDC preconditioner, we define an averaging operator E_D, which maps $\left(\widetilde{\boldsymbol{\Lambda}}_\Gamma, W_0\right)$, with generally discontinuous interface velocities, to the same space with continuous interface velocities: for any $w = (w_\Gamma, q_0) \in \left(\widetilde{\boldsymbol{\Lambda}}_\Gamma, W_0\right)$,

$$E_D\begin{bmatrix} w_\Gamma \\ q_0 \end{bmatrix} = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix}\begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix}\begin{bmatrix} w_\Gamma \\ q_0 \end{bmatrix} = \begin{bmatrix} E_{D,\Gamma} w_\Gamma \\ q_0 \end{bmatrix} \in \left(\widetilde{\boldsymbol{\Lambda}}_\Gamma, W_0\right),$$

where $E_{D,\Gamma} = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T$ averages the interface velocity with a properly defined weight.

The following lemma is the result of the $E_{D,\Gamma}$ for the elliptic problem, see (Tu & Wang, 2016, Lemma 5).

Lemma 5.1.7. *There exists a positive constant C, which is independent of H and h, such that*

$$|E_{D,\Gamma}\boldsymbol{\lambda}_\Gamma|^2_{\widetilde{S}_{\Gamma,E}} \leq C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\boldsymbol{\lambda}_\Gamma|^2_{\widetilde{S}_{\Gamma,E}}, \qquad \forall\ \boldsymbol{\lambda}_\Gamma \in \widetilde{\boldsymbol{\Lambda}}_\Gamma,$$

where $\gamma_{h,\tau}$ is a constant defined in (5.21).

Now, we prove the bound of the averaging operator E_D for the Stokes problem.

Lemma 5.1.8. *If $h_\kappa \tau_\kappa \approx C$, then there exists a positive constant C, which is independent of*

H and h, such that

$$|E_D w|^2_{\widetilde{S}} \leq C\frac{(1+\beta)^2}{\beta^2}\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 |w|^2_{\widetilde{S}} \qquad \forall\ w=(\boldsymbol{\lambda}_\Gamma, q_0)\in\left(\widetilde{\boldsymbol{\Lambda}}_{\Gamma,B}, W_0\right),$$

where β is the inf-sup stability constant, and $\gamma_{h,\tau}$ is a constant defined in (5.21).

Proof. For any vector $w=(\boldsymbol{\lambda}_\Gamma, q_0)\in\left(\widetilde{\boldsymbol{\Lambda}}_{\Gamma,B}, W_0\right)$, by Lemma 5.1.1, $\widetilde{R}^T_{D,\Gamma}\boldsymbol{\lambda}_\Gamma\in\widehat{\boldsymbol{\Lambda}}_{\Gamma,B}$. Thus, $E_{D,\Gamma}\boldsymbol{\lambda}_\Gamma=\widetilde{R}_\Gamma\widetilde{R}^T_{D,\Gamma}\boldsymbol{\lambda}_\Gamma\in\widetilde{\boldsymbol{\Lambda}}_{\Gamma,B}$.

From the definition of the $\widetilde{S}$-seminorm, we have

$$|E_D w|^2_{\widetilde{S}}=\left\|E_{D,\Gamma}\boldsymbol{\lambda}_\Gamma\right\|^2_{\widetilde{S}_\Gamma}=|\bar{R}_\Gamma\left(E_{D,\Gamma}\boldsymbol{\lambda}_\Gamma\right)|^2_{S_\Gamma}. \tag{5.42}$$

By Lemmas 5.1.6, 5.1.7, we have

$$\begin{aligned}
&|\bar{R}_\Gamma\left(E_{D,\Gamma}\boldsymbol{\lambda}_\Gamma\right)|^2_{S_\Gamma}\leq C\frac{(1+\beta)^2}{\beta^2}|\bar{R}_\Gamma\left(E_{D,\Gamma}\boldsymbol{\lambda}_\Gamma\right)|^2_{S_{\Gamma,E}}\\
&\leq C\frac{(1+\beta)^2}{\beta^2}\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\boldsymbol{\lambda}_\Gamma|^2_{\widetilde{S}_{\Gamma,E}}\leq C\frac{(1+\beta)^2}{\beta^2}\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\boldsymbol{\lambda}_\Gamma|^2_{\widetilde{S}_\Gamma}.
\end{aligned}$$

Then, we have

$|E_D w|^2_{\widetilde{S}}\leq C\frac{(1+\beta)^2}{\beta^2}\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|\boldsymbol{\lambda}_\Gamma|^2_{\widetilde{S}_\Gamma}\leq C\frac{(1+\beta)^2}{\beta^2}\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2|w|^2_{\widetilde{S}}.$

□

Remark. In (Tu & Wang, 2016), for the elliptic problems, the algorithm is proved to be scalable with the choices of the stabilization constant τ_κ to be 1 and $\frac{1}{h_\kappa}$, see Lemma 5.1.7 and (5.21). For the Stokes problem, we require $\tau_\kappa h_\kappa\approx C$, which excludes the case with $\tau_\kappa=1$. This condition is required in the proof of (5.38). In our numerical experiments, we test three choices of τ_κ as in (Tu & Wang, 2016), namely $\tau_\kappa=1,\ \frac{1}{h_\kappa},\ \frac{1}{h^2_\kappa}$. With all choices of τ_κ, the BDDC algorithms perform similarly to the elliptic cases, but our theorem is only valid for the choice with $\tau_\kappa=\frac{1}{h_\kappa}$.

5.1.6 Conditioner number estimate for the BDDC preconditioner

We are now ready to formulate and prove our main result; it follows as in the proof of (Li & Widlund, 2006a, Theorem 6.7) using Lemma 5.1.8. Also see the proof of (Mandel et al., 2005, Theorem 25), (Tu, 2007d, Lemma 4.6), and (Tu, 2007c, Lemma 4.7).

Theorem 5.1.9. *The preconditioned operator $M^{-1}\widehat{S}$ is symmetric, positive definite with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\widehat{S}}$ on the space $\left(\widehat{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right)$. If $h_\kappa\tau_\kappa \approx C$, the condition number of $M^{-1}\widehat{S}$ is bounded from below by 1, and from above by $C\frac{(1+\beta)^2}{\beta^2}\gamma_{h,\tau}\left(1+\log\left(\frac{H}{h}\right)\right)^2$, where C is a constant which is independent of H and h. $\gamma_{h,\tau}$ is a constant defined in (5.21) and β is the inf-sup stability constant.*

Proof:

It is enough to prove that, for any $u \in \left(\widehat{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right)$ with $u_\Gamma \neq 0$,

$$u^T M u \leq u^T \widehat{S} u \leq C\frac{(1+\beta)^2}{\beta^2}\gamma_{h,\tau}\left(1+\log(H/h)\right)^2 u^T M u,$$

Lower bound: Let

$$w = \left(\widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D\right)^{-1} u \in \left(\widehat{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right). \tag{5.43}$$

Using the properties $\widetilde{R}^T\widetilde{R}_D = \widetilde{R}_D^T\widetilde{R} = I$ and (5.43), we have,

$$\begin{aligned}
& u^T M u = u^T\left(\widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D\right)^{-1} u = u^T w \\
= \;& u^T \widetilde{R}^T \widetilde{S} \widetilde{S}^{-1} \widetilde{R}_D w = \left\langle \widetilde{R}u, \widetilde{S}^{-1}\widetilde{R}_D w\right\rangle_{\widetilde{S}} \\
\leq \;& \left\langle \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{S}}^{1/2} \left\langle \widetilde{S}^{-1}\widetilde{R}_D w, \widetilde{S}^{-1}\widetilde{R}_D w\right\rangle_{\widetilde{S}}^{1/2} \\
= \;& \left(u^T \widetilde{R}^T \widetilde{S} \widetilde{R} u\right)^{1/2} \left(w^T \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{S} \widetilde{S}^{-1} \widetilde{R}_D w\right)^{1/2} \\
= \;& \left(u^T \widetilde{R}^T \widetilde{S} \widetilde{R} u\right)^{1/2} \left(u^T M u\right)^{1/2}.
\end{aligned}$$

We obtain

$$u^T M u \leq u^T \widehat{S} u,$$

by canceling a common factor and squaring.

Upper bound: Using the definition of w in (5.43), the Cauchy-Schwarz inequality, and Lemma 5.1.8, we obtain the upper bound:

$$\begin{aligned}
& u^T \widehat{S} u = u^T \widetilde{R}^T \widetilde{S} \widetilde{R} \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D w \\
= \;& \left\langle \widetilde{R}u, E_D \widetilde{S}^{-1} \widetilde{R}_D w \right\rangle_{\widetilde{S}} \\
\leq \;& \left\langle \widetilde{R}u, \widetilde{R}u \right\rangle_{\widetilde{S}}^{1/2} \left\langle E_D \widetilde{S}^{-1} \widetilde{R}_D w, E_D \widetilde{S}^{-1} \widetilde{R}_D w \right\rangle_{\widetilde{S}}^{1/2} \\
\leq \;& C \left\langle \widetilde{R}u, \widetilde{R}u \right\rangle_{\widetilde{S}}^{1/2} \frac{(1+\beta)}{\beta} \gamma_{h,\tau}^{1/2} \left(1+\log(H/h)\right) |\widetilde{S}^{-1} \widetilde{R}_D w|_{\widetilde{S}} \\
= \;& C \frac{(1+\beta)}{\beta} \gamma_{h,\tau}^{1/2} \left(1+\log(H/h)\right) \left(u^T \widetilde{R}^T \widetilde{S} \widetilde{R} u\right)^{1/2} \left(w^T \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{S} \widetilde{S}^{-1} \widetilde{R}_D w\right)^{1/2} \\
= \;& C \frac{(1+\beta)}{\beta} \gamma_{h,\tau}^{1/2} \left(1+\log(H/h)\right) \left(u^T \widehat{S} u\right)^{1/2} \left(u^T M u\right)^{1/2}.
\end{aligned}$$

Thus,

$$u^T \widehat{S} u \leq C \frac{(1+\beta)^2}{\beta^2} \gamma_{h,\tau} \left(1+\log(H/h)\right)^2 u^T M u.$$

□

5.1.7 Numerical Experiments

We have applied our BDDC algorithms to the model problem (5.1), where $\Omega = [0,1]^2$. Zero Dirichlet boundary conditions are used. The right-hand side function $\boldsymbol{f}$ is chosen such that the exact solution is

$$\mathbf{u} = \begin{bmatrix} \sin^3(\pi x) \sin^2(\pi y) \cos(\pi y) \\ \\ -\sin^2(\pi x) \sin^3(\pi y) \cos(\pi x) \end{bmatrix} \quad \text{and} \quad p = x^2 - y^2.$$

We decompose the unit square into $N \times N$ subdomains with the sidelength $H = 1/N$. Equation (5.1) is discretized, in each subdomain, by the pth-order HDG method with a element diameter h. The preconditioned conjugate gradient iteration is stopped when the relative l_2-norm of the residual has been reduced by a factor of 10^6.

Table 5.1: Performance of solving (5.19) with HDG discretization ($tau = 1$)

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	4.13	10	4.46	12	12.12	14
	8×8	5.03	13	8.34	17	11.27	20
	16×16	4.88	13	9.87	20	13.16	24
	24×24	5.04	13	10.26	20	13.67	24
	32×32	4.94	12	10.23	19	13.77	24
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	2.49	9	5.86	13	8.32	17
	8	5.03	13	8.34	17	11.27	20
	16	7.49	15	11.28	20	17.51	24
	24	9.12	17	13.22	21	19.83	25
	32	10.37	19	14.69	22	21.15	25

We consider three different choices of the stabilization constant τ_κ, namely $\tau_\kappa = 1$, $\tau_\kappa = \frac{1}{h_\kappa}$, and $\tau_\kappa = \frac{1}{h_\kappa^2}$ as those in (Tu & Wang, 2016) for elliptic problems. For each choice of τ_κ, we have carried out experiments to obtain iteration counts and condition number estimates. The performance of the algorithms for the Stokes problem with these three choices of τ_κ is similar to those for the elliptic problems. The experimental results for $\tau_\kappa = \frac{1}{h_\kappa}$ are fully consistent with our theory. Our theory does not apply to the cases of $\tau_\kappa = 1$ and $\tau_\kappa = \frac{1}{h_\kappa^2}$, as we point out in the *Remark*. We note that, for the choices of $\tau_\kappa = 1$, the algorithms work as good as $\tau_\kappa = \frac{1}{h_\kappa}$. As for the elliptic case, for $\tau = \frac{1}{h_\kappa^2}$, $\gamma_{h,\tau} \approx \frac{1}{h_\kappa}$ and the condition number is linearly increasing with the mesh refinement. The algorithm is not scalable anymore.

Table 5.2: Performance of solving (5.19) with HDG discretization ($tau = 1/h_\kappa$)

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	4.21	10	4.72	12	12.72	14
	8×8	5.12	12	8.81	17	11.52	20
	16×16	5.00	13	10.43	21	13.44	24
	24×24	5.14	13	10.83	20	13.96	25
	32×32	5.14	13	10.84	20	14.09	25
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	2.56	9	6.23	14	8.52	17
	8	5.12	12	8.81	17	11.52	20
	16	7.59	15	11.86	20	17.86	24
	24	9.22	17	13.86	22	20.32	25
	32	10.48	19	15.37	23	22.21	26

5.2 BDDC for Stokes Problem with WG Method

5.2.1 Introduction

Numerical solution of saddle point problems using non-overlapping domain decomposition methods have long been an active area of research; see, e.g., (Pavarino & Widlund, 2002; Li, 2005; Goldfeld et al., 2003; Dohrmann, 2007b; Li & Widlund, 2006a; Tu, 2007a, 2005; Li, 2005; Li & Tu, 2013; Tu & Li, 2014, 2013, 2015; Pavarino & Scacchi, 2016). The Balancing Domain Decomposition by Constraints (BDDC) algorithm is an advanced variant of the non-overlapping domain decomposition technique. It was first introduced by Dohrmann in 2003 (Dohrmann, 2003), and the theoretical analysis was later given by Mandel, Dohrmann (Mandel & Dohrmann, 2003). In this theoretical development, optimal condition number bound was obtained for the BBDC operators proposed for symmetric positive definite systems. Nonetheless, the variational form of the incompressible Stokes problem is a saddle point problem (Brezzi & Fortin, 1991), and the discretization by finite element methods lead

Table 5.3: Performance of solving (5.19) with HDG discretization ($tau = 1/h_\kappa^2$)

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	4.48	11	22.13	16	19.54	17
	8×8	5.36	13	36.61	30	27.04	29
	16×16	5.36	14	62.50	45	36.51	37
	24×24	5.40	14	87.78	50	47.51	41
	32×32	5.33	13	111.22	53	57.78	43
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	3.05	10	18.97	22	13.91	21
	8	5.36	13	36.61	30	27.04	29
	16	7.77	16	73.03	40	44.63	36
	24	9.38	18	111.36	46	60.93	43
	32	10.61	19	151.09	48	76.86	46

to symmetric indefinite matrices. Thus, the conventional theory usually fails to apply. In the first attempt to apply BDDC to the incompressible Stokes problem by Li and Widlund (Li & Widlund, 2006a), the approach via benign spaces was used to reduce the Stokes system to a symmetric positive definite problem, and optimal convergence result was obtained as for the elliptic case. However, this method was proposed and analyzed with discontinuous pressure approximation, and there is a big class of mixed finite element spaces featuring continuous pressure, e.g., the Taylor-Hood finite elements. Later, Li and Tu proposed a non-overlapping domain decomposition algorithm for continuous finite element pressure space, which was proved and numerically verified to be scalable(Li & Tu, 2013). Earlier, Šístek et al. applied a parallel BDDC pre-conditioner based on the corner constraints to the Stokes flow using mixed discretization by Taylor-Hood finite element(Šístek et al., 2011). They numerically demonstrated the promising speedup property of their BDDC pre-conditioner as applied to benchmark test problems of real-life relevance, even though optimal scalability was not achieved.

As the property of the discretized system to be solved is dependent on the numerical methods used, in this study, we design BDDC pre-conditioners for trending non-conforming finite element methods, in particular, the hybridizable discontinuous Galerkin (HDG) and weak Galerkin(WG) methods.

Non-overlapping domain decomposition have been widely studied and applied for solving large symmetric positive definite linear systems arising from the discretization of elliptic partial differential equations(Toselli & Widlund, 2005). The balancing domain decomposition by constraints (BDDC) algorithms are domain decomposition methods based on non-overlapping subdomain division. They represent an important class of iterative substructuring methods. This method was first introduced by Dohrmann(Dohrmann, 2003), and further analyzed in the elliptic case by Mandel, Dohrmann, and Tezaur (Mandel & Dohrmann, 2003; Mandel et al., 2005). In BDDC methods, a coarse problem is proposed across the interface formed by parts of the boundaries of at least two subdomains to enforce the primal continuity constraints. The primal variables, which will be the same across the interface for each iteration, include point constraints, edge or face average constraints, and for some applications constraints for first order moments. One advantage with such designed coarse problem is that the corresponding Schur complements are invertible. This method is a successful redesign of the balancing Neumann-Neumann (BNN) alogrithm in the same way as FETI-DP algorithms to the older one-level FETI. The BDDC algorithms have been extended to the second order elliptic problem with mixed and hybrid formulations, hybridizable discontinuous Galerkin (HDG) methods (Tu, 2005, 2007a; Tu & Wang, 2016) and the Stokes problem with standard finite element and HDG method (Li & Widlund, 2006a; Tu & Wang, 2017a).

In this work, a BDDC algorithm is developed for weak Galerkin discretization of the incompressible Stokes problem. The weak Galerkin (WG) methods are a class of nonconforming finite element methods, which were first introduced for second order elliptic problems by Wang and Ye(Wang & Ye, 2013). The idea of WG is to introduce weak functions and their

weak derivatives as distributions, which can be approximated by polynomials of different degrees on different support. For example, for second order elliptic problems, weak functions have the form of $v = \{v_0, v_b\}$, where v_0 is defined inside each element and v_b is defined on the boundary of the element. v_0 and v_b can both be approximated by polynomials. The gradient operator is approximated by a *weak gradient* operator, which is further approximated by polynomials. These weakly defined functions and derivatives make the WG methods highly flexible in terms of approximating functions and finite element partition of the domain. The same *weak* concepts have been extended to other differential operators such as *divergence* and *curl,* which appears in applications like Stokes (Wang & Ye, 2016)and Maxwell(Mu et al., 2015c) equations respectively.

As most finite element methods, the WG methods result in a large number of degrees of freedom and therefore require solving large linear systems with condition number deteriorating with the refinement of the mesh. Efficient fast solvers for the resulting linear system are necessary. However, relatively few attempts on designing fast solvers for the WG methods can be found in the literature; see (Chen et al., 2015). An effective implementation of WG method is to reduce the unknown variables to those associated with element boundaries through a Schur-complement approach. It can be further reduced to the subdomain interface. The interface problem can then be solved using the conjugate gradient method preconditioned with a BDDC algorithm. In addition to point constraints, it is also necessary to impose edge or face average constraints across the interface. By a change of variable(Li & Widlund, 2006b; Klawonn & Widlund, 2006), the primal constraint on edge or face average can be converted to an explicit variable. The reduced system for the primal variables will be the coarse problem to solve. The BDDC preconditioner can be built based on such designed coarse problem, and thus be used as a preconditioner for the conjugate gradient method.

In a recent study(Tu & Wang, 2016), the authors proved the condition number bound for elliptic problems with hybridizable discontinuous Galerkin discretizations using its spectral equivalence with that of a hybridized RT method, which was previously studied by Tu(Tu,

2007a). Later this result was extended to the same problem with WG discretization by drawing connections between these two methods(Tu & Wang, 2017c). In this work, a BDDC algorithm is further developed for weak Galerkin discretization of the incompressible Stokes problem in a similar way to (Li & Widlund, 2006a). The preconditioned Stokes problem such designed is positive definite when restricted to certain *benign subspaces*, and its iterates stay in this subspace. It can be proved that the condition number bound is as strong as for the elliptic case.

5.2.2 A Stokes problem and its weak Galerkin Discretization

We consider the primary velocity-pressure formulation for the Stokes problem on a bounded polygonal domain Ω, in two dimenisons ($n=2$), or three dimensions ($n=3$), with a Dirichlet boundary condition:

$$\begin{cases} -\triangle u + \nabla p = f & \text{in} \quad \Omega, \\ \nabla \cdot u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{5.44}$$

where $f \in \left[L^2(\Omega)\right]^n$, and $g \in \left[H^{1/2}(\partial\Omega)\right]^n$. Without loss of generality, we assume that $g=0$. The weak form in the primary velocity-pressure formulation for the Stokes problem seeks $u \in \left[H_0^1(\Omega)\right]^n$ and $p \in L_0^2(\Omega)$ such that

$$\begin{cases} (\nabla u, \nabla v) - (\nabla \cdot v, p) &= (f, v) & \forall v \in \left[H_0^1(\Omega)\right]^n, \\ (\nabla \cdot u, q) &= 0 & \forall q \in L_0^2(\Omega). \end{cases} \tag{5.45}$$

The idea of weak Galerkin finite element scheme is to substitute the standard function and differential operators with the weakly defined counterparts. A weak function over the domain D is defined as $v = \{v_0, v_b\}$ such that $v_0 \in L^2(D)$ and $v_b \in H^{1/2}(\partial D)$. The v_0 part represents the value of v in the interior of D, while the v_b part represents the value of v

on the boundary of D. Note that v_b does not bind itself with v_0 from the definition. In essence, weak functions relax the continuity property of the standard functions, thus to offer more flexibility in terms of variable representation. We denote by $\mathcal{W}(D)$ the space of weak functions over the domain D

$$\mathcal{W}(D) = \left\{ v = \{v_0, v_b\} : v_0 \in L^2(D), v_b \in H^{1/2}(\partial D) \right\},$$

and the relevant vector-valued weak function space by

$$[\mathcal{W}(D)]^n = \left\{ \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in \left[L^2(D)\right]^n, \mathbf{v}_b \in \left[H^{1/2}(\partial D)\right]^n \right\},$$

and

$$[W(D)]^n = \left\{ \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in \left[L^2(D)\right]^n, \mathbf{v}_b \cdot \mathbf{n} \in H^{-1/2}(\partial D) \right\}.$$

The space of weak gradient or divergence operators will be defined as the dual space of appropriate Hilbert space, in similar manner as the dual of $\left[L^2(D)\right]^n$ can be identified with itself by using the L^2 inner product as the action of the linear functionals.

Definition 5.2.1. *For any $\mathbf{v} \in [\mathcal{W}(D)]^n$, the weak gradient of $\mathbf{v}$ is defined as the linear functional $\nabla_w \mathbf{v}$ in the dual space of $H(div; D)$ whose action on each $\mathbf{q} \in H(div; D)$ is given by*

$$(\nabla_w \mathbf{v}, \mathbf{q})_D = -(\mathbf{v}_0, \nabla \cdot \mathbf{q})_D + \langle \mathbf{v}_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial D},$$

where $\mathbf{n}$ is the outward normal direction to $\partial\Omega$.

Definition 5.2.2. *For any $\mathbf{v} \in [W(D)]^n$, the weak divergence of $\mathbf{v}$ is defined as the linear functional $\nabla_w \cdot \mathbf{v}$ in the dual space of $H^1(D)$ whose action on each $\varphi \in H^1(D)$ is given by*

$$(\nabla_w \cdot \mathbf{v}, \varphi)_D = -(\mathbf{v}_0, \nabla \varphi)_D + \langle \mathbf{v}_b \cdot \mathbf{n}, \varphi \rangle_{\partial D},$$

where $\mathbf{n}$ is the outward normal direction to $\partial\Omega$.

Now, we are in a position to introduce the weak Galerkin finite element algorithm. First, we introduce the mesh of the domain, then we will define discontinuous weak Galerkin finite element spaces over the mesh. Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω, and the element in $\mathcal{T}_h$ denoted by K. For any $K \in \mathcal{T}_h$, we denote by h_K the diameter of K with $h = \max_{K\in\mathcal{T}_h} h_K$. Define $\mathcal{F}_h$ be the set of edges/faces of elements $K \in \mathcal{T}_h$. $\mathcal{F}_h^i$ and $\mathcal{F}_h^\partial$ are subsets of $\mathcal{F}_h$, which consists of domain interior and boundary edges, respectively. For any domain D, let $P_k(D)$ be the space of polynomials of degree $\leq k$ on D. Define the weak Galerkin finite element spaces for the velocity variable associated with $\mathcal{T}_h$ as follows:

$$V_k = \left\{ v = \{v_0, v_b\} : \{v_0, v_b\}|_K \in [P_k(K)]^n \times [P_{k-1}(e)]^n, \forall K \in \mathcal{T}_h, e \subset \partial K \right\}.$$

Note that a function $v \in V_k$ has a single value v_b on each edge $e \in \mathcal{F}_h$. The subspace of V_k with vanishing boundary values on $\partial\Omega$ is denoted by

$$V_k^0 = \{ v = \{v_0, v_b\} \in V_k : v_b = 0 \,\text{on}\, \partial\Omega \}.$$

We denote the standard piecewise polynomial finite element space by

$$\mathring{V}_k = \left\{ v : v|_K \in [P_k(K)]^n, \forall K \in \mathcal{T}_h \right\},$$

and a relevant matrix polynomial function space by

$$\mathbf{Q}_{k-1} = \left\{ \mathbf{v} : \mathbf{v}|_K \in [P_{k-1}(K)]^{n\times n}, \forall K \in \mathcal{T}_h \right\}.$$

For the pressure variable, define the following finite element space

$$W_{k-1} = \left\{ q : q \in L_0^2(\Omega), q|_K \in P_{k-1}(K) \right\}.$$

Denote the discrete weak gradient operator by $\nabla_{w,k-1}$, and the discrete weak divergence

operator by $\left(\nabla_{w,k-1}\cdot\right)$, respectively. On the finite element space V_k, they are defined as follows: for $v=\{v_0,v_b\}\in V_k$, on each element $K\in\mathcal{T}_h$, $\nabla_{w,k-1}v\mid_K\in[P_{k-1}(K)]^{n\times n}$ and $\nabla_{w,k-1}\cdot v\mid_K\in P_{k-1}(K)$ are the unique solutions of the following equations, respectively,

$$\left(\nabla_{w,k-1}v\mid_K,\mathbf{q}\right)_K=-\left(v_{0,K},\nabla\cdot\mathbf{q}\right)_K+\left\langle v_{b,K},\mathbf{q}\cdot\mathbf{n}\right\rangle_{\partial K},\qquad\forall\mathbf{q}\in[P_{k-1}(K)]^{n\times n},$$

$$\left(\nabla_{w,k-1}\cdot v\mid_K,\varphi\right)_K=-\left(v_{0,K},\nabla\varphi\right)_K+\left\langle v_{b,K}\cdot\mathbf{n},\varphi\right\rangle_{\partial K},\qquad\forall\varphi\in P_{k-1}(K),$$

where $v_{0,K}$ and $v_{b,K}$ are the restrictions of v_0 and v_b to K, respectively, $(u,w)_K=\int_K uwdx$, and $\langle u,w\rangle_{\partial K}=\int_{\partial K}uwds$. To simplify the notation, we shall drop the subscript $k-1$ in the notation $\nabla_{w,k-1}$ and $\left(\nabla_{w,k-1}\cdot\right)$ for the discrete weak gradient and the discrete weak divergence operator. We denote the L^2 inner product over the triangulation as a summation over each element of the triangulation, for example,$(\nabla_w u,\nabla_w w)_{\mathcal{T}_h}=\sum_{K\in\mathcal{T}_h}(\nabla_w u,\nabla_w w)_K$, $(\nabla_w\cdot v,q)_{\mathcal{T}_h}=\sum_{K\in\mathcal{T}_h}(\nabla_w\cdot v,q)_K$.

Denote by Q_0 the L^2 projection from $\left[L^2(K)\right]^n$ onto $[P_k(K)]^n$, and denote by Q_b the L^2 projection from $\left[L^2(e)\right]^n$ onto $[P_{k-1}(e)]^n$, for $e\in\mathcal{F}_h$. And we write the correponding projection operator for the weak function as $Q_h=\{Q_0,Q_b\}$. Next, we introduct three bilinear forms, which will be used in the weak Galerkin finite element discretization for the Stokes equation as below,

$$\begin{aligned}s(v,w)&=\textstyle\sum_{K\in\mathcal{T}_h}h_K^{-1}\langle Q_bv_0-v_b,Q_bw_0-w_b\rangle_{\partial K},\\ a(v,w)&=(\nabla_w v,\nabla_w w)_{\mathcal{T}_h}+s(v,w),\\ b(v,q)&=(\nabla_w\cdot v,q)_{\mathcal{T}_h}.\end{aligned}\tag{5.46}$$

The discrete problem resulting from the WG discretization can then be written as: find $u_h=\{u_0,u_b\}\in V_k^0$ and $p_h\in W_{k-1}$ such that

$$\begin{cases}a(u_h,v)-b(v,p_h)=(f,v_0),&\forall v=\{v_0,v_b\}\in V_k^0,\\ b(u_h,q)=0,&\forall q\in W_{k-1}.\end{cases}$$

We introduce the following operators: $A : V_k^0 \to V_k^0$, $B : V_k^0 \to W_{k-1}$, by

$$(Au_h, v) = a(u_h, v)\,,\ (Bu_h, q) = -b(u_h, q)\,. \tag{5.47}$$

Using these operators, the matrix form of the weak Galerkin scheme can be represented as

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

At element level, for each K, given the edge component v_b of the velocity and the pressure p, the interior component v_0 of the velocity can be uniquely determined. Namely, v_0 can be eliminated in each element independently. We thus obtain the reduced system of v_b and p only with considerable smaller size but different sparsity pattern as below

$$\begin{bmatrix} A_{uu} & B_{pu}^T \\ B_{pu} & C_{pp} \end{bmatrix} \begin{bmatrix} u_{h,b} \\ p_h \end{bmatrix} = \begin{bmatrix} f_{u_b} \\ f_p \end{bmatrix}.$$

Hereafter, we will work with the reduced system such obtained.

5.2.3 Reduced Subdomain Interface Problem

We decompose Ω into N nonoverlapping subdomain Ω_i with diameters H_i, $i = 1,\ldots,N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such elements forming an individual sudomain is uniformly bounded. We define edges/faces as open sets shared by two subdomains. Two nodes belong to the same face when they are associated with the same pair of subdomains. Let Γ be the interface between the subdomains. The set of the interface nodes $\mathit{\Gamma}_h$ is defined as $\mathit{\Gamma}_h := \left(\cup_{i\neq j}\partial\Omega_{i,h} \cap \partial\Omega_{j,h}\right) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is that of $\partial\Omega$. We assume the triangulation of each subdomain is quasi-uniform.

We decompose the discrete velocity and pressure spaces V_k and W_{k-1} into:

$$V = V_I \oplus \hat{V}_\Gamma, \qquad W = W_I \oplus W_0.$$

We drop the subscripts k and $k-1$ for simplicity. Herein, V_I and W_I are products of subdomain interior velocity spaces $V_I^{(i)}$ and subdomain interior pressure spaces $W_I^{(i)}$, respectively; i.e.,

$$V_I = \prod_{i=1}^{N} V_I^{(i)}, \qquad W_I = \prod_{i=1}^{N} W_I^{(i)}.$$

The elements of $V_I^{(i)}$ are supported in the subdomain Ω_i and vanishes on its interface Γ_i, while the elements of $W_I^{(i)}$ are restrictions of the pressure variables to Ω_i which satisfy $\int_{\Omega_i} p_I^{(i)} = 0$. $\widehat{V}_\Gamma$ is the subspace of edge functions on Γ, and W_0 is the subspace of W with constant values $p_0^{(i)}$ in the subdomain Ω_i that satisfy $\sum_{i=1}^{N} p_0^{(i)} m(\Omega_i) = 0$, where $m(\Omega_i)$ is the measure of the subdomain Ω_i.

We denote the space of interface edge velocity variables of the subdomain Ω_i by $V_\Gamma^{(i)}$, and the associated product space by $V_\Gamma = \prod_{i=1}^{N} V_\Gamma^{(i)}$; generally edge functions in V_Γ are discontinuous across the interface. We define the restriction operators $R_\Gamma^{(i)} : \hat{V}_\Gamma \to V_\Gamma^{(i)}$ to be an operator which maps functions in the continuous global interface edge variable space $\hat{V}_\Gamma$ to the subdomain component space $V_\Gamma^{(i)}$. Also, $R_\Gamma : \hat{V}_\Gamma \to V_\Gamma$ is the direct sum of $R_\Gamma^{(i)}$. We denote the spaces of the right-hand-side interior load vectors f_I and interface load vectors f_Γ by F_I and F_Γ, respectively. Similar notation conventions apply to the spaces $\widetilde{F}_\Gamma$, $\widehat{F}_\Gamma$, $\widehat{F}_\Pi$, $F_\Delta^{(i)}$, $F_\Gamma^{(i)}$, and F_0. We will use them in what follows without further explanation.

With the decomposition of the solution space, the global Stokes problem can be written as follows: find $(u_I, p_I, u_\Gamma, p_0) \in \left(V_I, W_I, \widehat{V}_\Gamma, W_0\right)$ such that

$$\begin{bmatrix} A_{II} & B_{II}^T & \widehat{A}_{\Gamma I}^T & 0 \\ B_{II} & C_{II} & \widehat{B}_{\Gamma I}^T & 0 \\ \widehat{A}_{\Gamma I} & \widehat{B}_{\Gamma I} & \widehat{A}_{\Gamma\Gamma} & \widehat{B}_{0\Gamma}^T \\ 0 & 0 & \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} u_I \\ p_I \\ u_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} f_I \\ f_{p_I} \\ f_\Gamma \\ 0 \end{bmatrix}. \tag{5.48}$$

For each subdomain problem, the dimension of the null space is one, and this corresponds to the undetermined degree of freedom of the mean pressure for each subdomain. For this reason, the bottom right block, which corresponds to the mean pressure from each subdomain, is zero. The lower left block in (5.48) is zero, because the bilinear form $b(u_h, \varphi)$ does not explicitly relate to u_I and p_I for any $u_h \in V_k^0$ and $\varphi \in W_0$. The leading two-by-two block of the matrix above can be rewritten into a block diagonal form with each block corresponding to an independent subdomain problem. And the global problem can be assembled from the constituent subdomain problems, as below

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)T} & \widehat{A}_{\Gamma I}^{(i)T} & 0 \\ B_{II}^{(i)} & C_{II}^{(i)} & \widehat{B}_{\Gamma I}^{(i)T} & 0 \\ \widehat{A}_{\Gamma I}^{(i)} & \widehat{B}_{\Gamma I}^{(i)} & \widehat{A}_{\Gamma\Gamma}^{(i)} & \widehat{B}_{0\Gamma}^{(i)T} \\ 0 & 0 & \widehat{B}_{0\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ p_I^{(i)} \\ u_\Gamma^{(i)} \\ p_0^{(i)} \end{bmatrix} = \begin{bmatrix} f_I^{(i)} \\ f_{p_I}^{(i)} \\ f_\Gamma^{(i)} \\ 0 \end{bmatrix}. \tag{5.49}$$

We can eliminate the subdomain interior variables $u_I^{(i)}$ and $p_I^{(i)}$ in each subdomain independently, and assemble the global interface problem from the subdomain interface problems.

Definition 5.2.3. *(Schur complement of the Stokes problem) Define the subdomain Schur complement $S_\Gamma^{(i)}$ for the Stokes problem as follows: given $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$, determine $S_\Gamma^{(i)} u_\Gamma^{(i)} \in F_\Gamma^{(i)}$ such that*

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)T} & A_{\Gamma I}^{(i)T} \\ B_{II}^{(i)} & C_{II}^{(i)} & B_{\Gamma I}^{(i)T} \\ A_{\Gamma I}^{(i)} & B_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ p_I^{(i)} \\ u_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ S_\Gamma^{(i)} u_\Gamma^{(i)} \end{bmatrix}. \tag{5.50}$$

The global interface problem can then be written as: to find $(u_\Gamma, p_0) \in \left(\widehat{V}_\Gamma, W_0\right)$, such that

$$\widehat{S} \begin{bmatrix} u_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} g_\Gamma \\ 0 \end{bmatrix}, \tag{5.51}$$

where

$\widehat{S} = \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix}$, $\widehat{S}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)T} S_\Gamma^{(i)} R_\Gamma^{(i)}$, $\widehat{B}_{0\Gamma} = \sum_{i=1}^N B_{0\Gamma}^{(i)} R_\Gamma^{(i)}$, and

$$g_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)T} \left\{ f_\Gamma^{(i)} - \begin{bmatrix} A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)T} \\ B_{II}^{(i)} & C_{II}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} f_I^{(i)} \\ f_{pI}^{(i)} \end{bmatrix} \right\}.$$

The operator $\widehat{S}_\Gamma$ is symmetric positive definite, because of the Dirichlet boundary conditions on $\partial\Omega$ and the primal continuity constraints defined on the interface. Let the operator of the global interface problem be denoted by $\widehat{S}$. Note that $\widehat{S}$ is symmetric indefinite. In what follows, we will propose a BDDC preconditioner, and show that the preconditioned operator is positive definite when restricted to a proper subspace. A preconditioned conjugate gradient method can then be used to solve the global interface problem.

5.2.4 The BDDC Preconditioner

The BDDC (Balancing Domain Decomposition by Constraints) algorithm is a variant of the two-level Neumann-Neumann type preconditioner. It was first introduced and analyzed by Dohrmann, Mandel, and Tezaur (Dohrmann, 2003; Mandel & Dohrmann, 2003; Mandel et al., 2005) for standard finite element discretization of elliptic problems. The BBDC preconditioner consists of local inexact solvers for the subdomain problems and the artistically designed global coarse-level problem. The coarse level problem is assembled from primal variables, such as edge/face averages across the interface on which the continuity constraints are enforced. In contrast to earlier versions of balancing Neumann-Neumann methods without coarse level problems, the BDDC methods do not need to solve singular systems and the

algorithms demonstrate good scalability for parallel computation.

In order to introduce the BDDC preconditioner, we first introduce a partially assembled interface space $\widetilde{V}_\Gamma$ by

$$\widetilde{V}_\Gamma = \widehat{V}_\Pi \oplus V_\Delta = \widehat{V}_\Pi \oplus \left(\prod_{i=1}^{N} V_\Delta^{(i)} \right).$$

Here, $\hat{V}_\Pi$ is the continuous, coarse level, primal interface edge velocity space. The variables in this space are called the primal unknowns, and each primal unknown is shared by the adjacent subdomains. The remaining interface velocity variables live in the complimentary dual space V_Δ . This space is the direct sum of the $V_\Delta^{(i)}$, which are spanned by basis functions with vanishing value at the primal degrees of freedom. The functions in V_Δ are generally discontinuous. Thus, in the space $\tilde{V}_\Gamma$, we relax the continuity constraints across the interface at the dual variables but retain the continuity at the primal variables, which makes all the component linear systems in the preconditioner nonsingular.

We require that $\int_{\partial\Omega_i} u_\Delta^{(i)} \cdot \mathbf{n}_i = 0$, for all the dual interface velocity variables $u_\Delta^{(i)} \in V_\Delta^{(i)}$, with $\mathbf{n}_i$ the unit outward normal of $\partial\Omega_i$; see(Li & Widlund, 2006a; Tu, 2005). We will refer to this assumption as the *divergence free constraint* for the dual velocity variables. In order to satisfy this constraint, we choose the primal variables which are spanned by subdomain interface edge/face basis functions with constant values on these edges/faces for two/three dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit; see (Li & Widlund, 2006b; Klawonn & Widlund, 2006). The dual space is correspondingly spanned by the remaining interface degrees of freedom with zero average values over the interface edge/face. This constraint is critical to the design of the preconditioner, as we will see more details in Subsection 5.2.5.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)} : \widetilde{V}_\Gamma \to V_\Gamma^{(i)}$ restricts functions in the space $\widetilde{V}_\Gamma$ to the components $V_\Gamma^{(i)}$ of the subdomain Ω_i. $\overline{R}_\Gamma : \widetilde{V}_\Gamma \to V_\Gamma$ is the direct sum of $\overline{R}_\Gamma^{(i)}$. $R_\Delta^{(i)} : \widehat{V}_\Gamma \to V_\Delta^{(i)}$ maps the functions from $\widehat{V}_\Gamma$ to $V_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi} : \widehat{V}_\Gamma \to \widehat{V}_\Pi$ is a restriction operator

from $\widehat{V}_\Gamma$ to its subspace $\widehat{V}_\Pi$. $\widetilde{R}_\Gamma : \widehat{V}_\Gamma \to \widetilde{V}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define the positive scaling factor $\delta_i^\dagger(x)$ as follows:

$$\delta_i^\dagger(x) = \frac{1}{card(\mathcal{I}_x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{I}_x$ is the set of indices of the subdomains that have x on their boundaries, and $card(\mathcal{I}_x)$ counts the number of the subdomain boundaries to which x belongs. It is clear that $\delta_i^\dagger(x)$'s provide a partition of unity, i.e., $\sum_{i\in\mathcal{I}_x}\delta_i^\dagger(x) = 1$, for any $x \in \Gamma_h$. We note that $\delta_i^\dagger(x)$ is constant on each edge. Multiplying each row of $R_\Delta^{(i)}$ with the scaling factor gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{D,\Delta}^{(i)}$.

The Schur complement $\widetilde{S}_\Gamma$ defined on the partially assembled interface velocity space $\widetilde{V}_\Gamma$ can be represented as follows: given $u_\Gamma \in \widetilde{V}_\Gamma$, $\widetilde{S}_\Gamma u_\Gamma \in \widetilde{F}_\Gamma$ satisfies

$$\begin{bmatrix} A_{II}^{(1)} & B_{II}^{(1)T} & A_{\Delta I}^{(1)T} & \cdots & \widetilde{A}_{\Pi I}^{(1)T} \\ B_{II}^{(1)} & C_{II}^{(1)} & B_{\Delta I}^{(1)T} & \cdots & \widetilde{B}_{\Pi I}^{(1)T} \\ A_{\Delta I}^{(1)} & B_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & \cdots & \widetilde{A}_{\Pi\Delta}^{(1)T} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{A}_{\Pi I}^{(1)} & \widetilde{B}_{\Pi I}^{(1)} & \widetilde{A}_{\Pi\Delta}^{(1)} & \cdots & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_I^{(1)} \\ p_I^{(1)} \\ u_\Delta^{(1)} \\ \vdots \\ u_\Pi \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ \left(\widetilde{S}_\Gamma u_\Gamma\right)_\Delta^{(1)} \\ \vdots \\ \left(\widetilde{S}_\Gamma u_\Gamma\right)_\Pi \end{bmatrix}.$$

Hereinabove, $\widetilde{A}_{\Pi\Pi} = \sum_{i=1}^N R_\Pi^{(i)T} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)}$, $\widetilde{A}_{\Pi I}^{(i)} = R_\Pi^{(i)T} A_{\Pi I}^{(i)}$, $\widetilde{A}_{\Pi\Delta}^{(i)} = R_\Pi^{(i)T} A_{\Pi\Delta}^{(i)}$, and $\widetilde{B}_{\Pi I}^{(i)} = R_\Pi^{(i)T} B_{\Pi I}^{(i)}$.

Based on this definition, we can also obtain $\widetilde{S}_\Gamma$ from subdomain Schur complements $S_\Gamma^{(i)}$ by assembling with respect to the global degrees of freedom of the primal interface velocities, i.e.,

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma. \tag{5.52}$$

Here, we denote the direct sum of $S_\Gamma^{(i)}$ by S_Γ. The global interface Schur operator $\widehat{S}_\Gamma$ on the continuous interface velocity space $\widehat{V}_\Gamma$ can be obtained by further assembling with respect

to the dual interface variables, i.e.,

$$\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma = R_\Gamma^T S_\Gamma R_\Gamma. \tag{5.53}$$

Correspondingly, we define an operator $\widetilde{B}_{0\Gamma}$, which maps the partially assembled interface velocity space $\widetilde{V}_\Gamma$ into F_0, the space of right-hand sides corresponding to W_0. $\widetilde{B}_{0\Gamma}$ can be obtained from the subdomain operators $B_{0\Gamma}^{(i)}$ by assembling with respect to the primal interface velocity part, i.e., $\widetilde{B}_{0\Gamma} = \sum_{i=1}^N B_{0\Gamma}^{(i)} \overline{R}_\Gamma^{(i)}$. Similarly, the operator $\widehat{B}_{0\Gamma}$ can be obtained from the partially assembled operator $\widetilde{B}_{0\Gamma}$ by further assembling with respect to the dual interface velocity variables on the subdomain interfaces, i.e., $\widehat{B}_{0\Gamma} = \widetilde{B}_{0\Gamma} \widetilde{R}_\Gamma$.

The preconditioner for solving the global interface Stokes problem is

$$M^{-1} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D,$$

where

$$\widetilde{R}_D = \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix}, \qquad \widetilde{S} = \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix}. \tag{5.54}$$

Note that $\widetilde{R}_{D,\Gamma}$ is of full rank and that the preconditioner is nonsingular. The preconditioned BDDC algorithm is then of the form: to find $(u_\Gamma, p_0) \in \left(\hat{V}_\Gamma, W_0\right)$, such that

$$M^{-1} \hat{S} \begin{bmatrix} u_\Gamma \\ p_0 \end{bmatrix} = M^{-1} \begin{bmatrix} g_\Gamma \\ 0 \end{bmatrix}.$$

5.2.5 Some Auxiliary Results

We adopt the convention that C denotes a generic constant independent of the mesh size h and subdomain size H. In general, its value may vary at different instances. For shape regular partition $\mathcal{T}_h$ as given in Appendix A, the trace inequality (1.5.3) and inverse inequality (1.5.4) hold; see details in (Wang & Ye, 2014).

We collect a few results of the weak Galerkin finite element scheme, which will be used

in our analysis of the BDDC preconditioner. Note that the discrete weak velocity function space V_k^0 is a normed linear space with a triple-bar norm given by(Wang & Ye, 2016)

$$|||v|||^2 = \sum_{K\in\mathcal{T}_h} \|\nabla_w v\|_K^2 + \sum_{K\in\mathcal{T}_h} h_K^{-1} \|Q_b v_0 - v_b\|_{\partial K}^2. \tag{5.55}$$

Lemma 5.2.1. *For the weak Galerkin scheme described in subsection 5.2.2, the following results hold:*

1. For any $v = \{v_0, v_b\} \in V_k$, we have $\sum_{T\in\mathcal{T}_h} \|\nabla v_0\|_T^2 \le C|||v|||^2$;
2. For any $v \in V_k^0$, $a(v,v) = |||v|||^2$;
3. For any $v, w \in V_k^0$, $|a(v,w)| \le |||v||| \, |||w|||$;
4. For any $v = \{v_0, v_b\} \in V_k^0$, $\rho \in W_{k-1}$, $|b(v,\rho)| \le C|||v||| \, \|\rho\|_{L^2}$;
5. For any $\rho \in W_{k-1}$, $\sup_{v\in V_k^0} \frac{b(v,\rho)}{|||v|||_w} \ge \beta \, \| \, \rho \, \|_{L^2}$, where β is positive constant independent of the mesh size h.

Proof. The first result is Lemma A.2 in (Wang & Ye, 2016); the second and third results give the coercivity and boundedness property of the bilinear form $a(\cdot,\cdot)$, which are proved in Lemma 5.1 in (Wang & Ye, 2016). The fourth result is the boundedness property of the bilinear form $b(\cdot,\cdot)$. This can be proved as follows.

$$\begin{aligned}
|b(v,\rho)| &= \left|\sum_{K\in\mathcal{T}_h} (\nabla_w \cdot v, \rho)_K\right| \\
&= \left|\sum_{K\in\mathcal{T}_h} (-(v_0, \nabla\rho)_K + \langle v_b \cdot n, \rho\rangle_{\partial K})\right| \\
&= \left|\sum_{K\in\mathcal{T}_h} ((\nabla\cdot v_0, \rho)_K - \langle (Q_b v_0 - v_b)\cdot n, \rho\rangle_{\partial K})\right| \\
&\le C\left(\sum_{K\in\mathcal{T}_h} \|\nabla v_0\|_{L^2(K)}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \|\rho\|_{L^2(K)}^2\right)^{1/2} \\
&\quad + C\left(\sum_{K\in\mathcal{T}_h} h_K^{-1} \|v_b - Q_b v_0\|_{L^2(\partial K)}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} h_k \|\rho\|_{L^2(\partial K)}^2\right)^{1/2} \\
&\le C|||v||| \, \|\rho\|_{L^2}
\end{aligned}$$

Note that we use the definition of weak divergence for the second equality, and itegration by parts for the third equality. We use the Cauchy-Schwarz inequality for the fourth

inequality, and part (1) of Lemma 5.2.1, definition of the triple-bar norm, eqn (5.55), and trace inequality, eqn (1.5.3), for the last inequality.

The last result is the discrete inf-sup condition, which is proved in Lemma 5.3(Wang & Ye, 2016). These results also hold for the subdmain Ω_i. If follows that the weak Galerkin scheme is well-posed for the global interface problem and local subdomain problems.

□

We introduce several conceptual tools which will be useful in our analysis of the BDDC preconditioner.

Definition 5.2.4. *(Schur complement of the subdomain elliptic problem) The subdomain Schur complement for the elliptic problem, denoted by* $S_{\Gamma,E}^{(i)}$*, is defined as follows: given* $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$*, determine* $S_{\Gamma,E}^{(i)} u_\Gamma^{(i)} \in F_\Gamma^{(i)}$ *such that*

$$A^{(i)} \begin{bmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ S_{\Gamma,E}^{(i)} u_\Gamma^{(i)} \end{bmatrix},$$

where $A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{\Gamma I}^{(i)T} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}$.

Since the subdomain elliptic problem $A^{(i)}$ is symmetric positive definite(Wang & Ye, 2013), the Schur complement $S_{\Gamma,E}^{(i)}$ is also symmetric positive definite by the inertia of Schur complements(Li & Widlund, 2006b). Thus, we can define the norm $\left|u^{(i)}\right|_{A^{(i)}}^2 = u^{(i)T} A^{(i)} u^{(i)} = a\left(u^{(i)}, u^{(i)}\right)$, for all $u^{(i)} \in V^{(i)}$, and $\left|u_\Gamma^{(i)}\right|_{S_{\Gamma,E}^{(i)}}^2 = u_\Gamma^{(i)T} S_{\Gamma,E}^{(i)} u_\Gamma^{(i)}$, for all $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$. Similarly, the subdomain Schur complements for the Stokes problems, defined in (5.50), are symmetric, positive semi-definite(Li & Widlund, 2006a). They are singular for any subdomains with floating boundaries, by which we mean the boundary of the subdomain does not intersect with the global domain boundary $\partial\Omega$. Thus, we can define the $S_\Gamma^{(i)}-$ and $S_\Gamma-$ seminorms by $\left|u_\Gamma^{(i)}\right|_{S_\Gamma^{(i)}}^2 = u_\Gamma^{(i)T} S_\Gamma^{(i)} u_\Gamma^{(i)}$, for all $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$. It follows that $|u_\Gamma|_{S_\Gamma}^2 = u_\Gamma^T S_\Gamma u_\Gamma = \sum_{i=1}^N \left|u_\Gamma^{(i)}\right|_{S_\Gamma^{(i)}}^2$.

By definition, it is not hard to show by direct computation that

$$\left|u_\Gamma^{(i)}\right|^2_{S_{\Gamma,E}^{(i)}} = \left|u_H^{(i)}\right|^2_{A^{(i)}}, \text{ and } \left|u_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}} = \left|u_S^{(i)}\right|^2_{A^{(i)}},$$

where $u_H^{(i)} \in V^{(i)}$ and $u_S^{(i)} \in V^{(i)}$ are the harmonic and Stokes extension, respectively.

The fully and partially assembled global interface velocity operators $\widehat{S}_\Gamma$ and $\widetilde{S}_\Gamma$, given in (5.53) and (5.52), are both symmetric, positive definite because of the Dirichlet boundary conditions on $\partial\Omega$ and the adequacy of the primal continuity constraints for the divergence free condition. In similar way as before, we define the $\widehat{S}_\Gamma-$ and $\widetilde{S}_\Gamma-$norms on the spaces $\widehat{V}_\Gamma$ and $\widetilde{V}_\Gamma$, respectively, as below.

$$\|u_\Gamma\|^2_{\widehat{S}_\Gamma} = u_\Gamma^T \widehat{S}_\Gamma u_\Gamma = u_\Gamma^T R_\Gamma^T S_\Gamma R_\Gamma u_\Gamma = |R_\Gamma u_\Gamma|^2_{S_\Gamma} \qquad \forall u_\Gamma \in \widehat{V}_\Gamma,$$

$$\|u_\Gamma\|^2_{\widetilde{S}_\Gamma} = u_\Gamma^T \widetilde{S}_\Gamma u_\Gamma = u_\Gamma^T \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma u_\Gamma = \left|\overline{R}_\Gamma u_\Gamma\right|^2_{S_\Gamma} \qquad \forall u_\Gamma \in \widetilde{V}_\Gamma.$$

The global interface operator $\widehat{S}$ and $\widetilde{S}$, introduced in (5.51) and (5.54), are symmetric indefinite on the space $\widehat{V}_\Gamma \times W_0$ and $\widetilde{V}_\Gamma \times W_0$, respectively. However, when restricted to the proper subspaces, these operators can be positive semidefinite, and we can thus define a $\widehat{S}-$ and $\widetilde{S}-$seminorms on these subspaces. We call such subspaces as the benign subspaces, and denote them by $\widehat{V}_{\Gamma,B} \times W_0$ and $\widetilde{V}_{\Gamma,B} \times W_0$, respectively. Specifically, they can be defined as follows.

Definition 5.2.5. *(Benign subspaces)*

$$\widehat{V}_{\Gamma,B} = \left\{u_\Gamma \in \widehat{V}_\Gamma | \widehat{B}_{0\Gamma} u_\Gamma = 0\right\} \quad \textit{and} \quad \widetilde{V}_{\Gamma,B} = \left\{u_\Gamma \in \widetilde{V}_\Gamma | \widetilde{B}_{0\Gamma} u_\Gamma = 0\right\}.$$

If follows that we can define

$$|u|^2_{\widehat{S}} = u^T \widehat{S} u \qquad \forall u = (u_\Gamma, p_0) \in \widehat{V}_{\Gamma,B} \times W_0,$$

$$|u|^2_{\widetilde{S}} = u^T \widetilde{S} u \qquad \forall u = (u_\Gamma, p_0) \in \widetilde{V}_{\Gamma,B} \times W_0.$$

We can show by direct computation that the following facts hold.

$$|u|^2_{\widehat{S}} = \|u_\Gamma\|^2_{\widehat{S}_\Gamma} \qquad \forall u = (u_\Gamma, p_0) \in \widehat{V}_{\Gamma,B} \times W_0,$$

$$|u|^2_{\widetilde{S}} = \|u_\Gamma\|^2_{\widetilde{S}_\Gamma} \qquad \forall u = (u_\Gamma, p_0) \in \widetilde{V}_{\Gamma,B} \times W_0.$$

We denote the null space of the $\widehat{S}-$seminorm operator on the space $\widehat{V}_{\Gamma,B} \times W_0$ by $\widehat{Z}$. It is easy to see that this space is comprised of elements $u = (0, p_0) \in \widehat{V}_{\Gamma,B} \times W_0$. The induced normed space $\left(\widehat{V}_{\Gamma,B} \times W_0\right) / \widehat{Z}$ is called the quotient space, and two elements (v_Γ, p_0) and (w_Γ, q_0), belonging to this quotient space, are identified if $v_\Gamma = w_\Gamma$.

The following lemma is crucial to the analysis of the preconditioned BDDC operator. The proof can be found in (Li & Widlund, 2006a; Tu, 2005).

Lemma 5.2.2. *Under the divergence free constraint for the dual interface velocities, introduced in Subsection 5.2.4, we have* $\widetilde{R}_D^T u \in \widehat{V}_{\Gamma,B} \times W_0$ *for any* $u \in \widetilde{V}_{\Gamma,B} \times W_0$.

With the choice of the primal velocity continuity constraints of the BDDC algorithm, the preconditioned BDDC operator $M^{-1}\widehat{S}$ is positive definite on the quotient space, and correspondingly, we can use the preconditioned conjugate gradient method when the iterations are restricted to the quotient space. The design of the BDDC preconditioner and the result from lemma 5.2.2 garantee that the iterations of the preconditioned conjugate gradient method will stay in the quotient subspace if the initialization lies in the quotient subspace(Li & Widlund, 2006a).

Next we introduce two important extension operators for the trace over the subdomain boundary.

Definition 5.2.6. *(Discrete harmonic exention) The discrete harmonic extension of* $\gamma \in V_\Gamma^{(i)}$ *over the submain* Ω_i*, denoted by* $\mathcal{H}(\gamma) : V_\Gamma^{(i)} \to V^{(i)}$*, satisfies the following:*

$$\begin{cases} a(\mathcal{H}(\gamma), v) = 0 \qquad \forall v = \{v_0, v_b\} \in V_k^0(\Omega_i), \\ \mathcal{H}(\gamma)|_{\partial\Omega_i} = \gamma. \end{cases}$$

The bilinear form $a(\cdot,\cdot)$ *is defined in (5.46).*

Definition 5.2.7. *(Discrete Stokes extension) The discrete Stokes extension of* $\gamma \in V_\Gamma^{(i)}$ *over the subdomain* Ω_i*, denoted by* $\mathcal{S}(\gamma) : V_\Gamma^{(i)} \to V^{(i)}$*, satisfies the following:*

$$\begin{cases} a(\mathcal{S}(\gamma), v) - b(v, \mathcal{P}(\gamma)) = 0 \qquad \forall v = \{v_0, v_b\} \in V_k^0(\Omega_i), \\ b(\mathcal{S}(\gamma), q) = 0 \qquad \forall q \in W_{k-1}(\Omega_i), \\ \mathcal{S}(\gamma)|_{\partial\Omega_i} = \gamma, \end{cases}$$

where $\mathcal{P}(\gamma)$ *is the correponding pressure extension with zero mean value living in the space* $W_{k-1}(\Omega_i)$*. The bilinear forms* $a(\cdot,\cdot)$ *and* $b(\cdot,\cdot)$ *are defined in (5.46).*

The connection between harmonic/Stokes extensions and the Schur complements of the corresponding linear systems can be revealed as follows.

Remark. By definition, it is clear that

$$\left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}^2 = \inf_{u^{(i)} \in V^{(i)},\, u^{(i)}|_{\partial\Omega_i} = u_\Gamma^{(i)}} \left|u^{(i)}\right|_{A^{(i)}}^2,$$

and that

$$\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}^2 = \inf_{u^{(i)} \in V^{(i)},\, u^{(i)}|_{\partial\Omega_i} = u_\Gamma^{(i)},\, B^{(i)}u^{(i)} = 0} \left|u^{(i)}\right|_{A^{(i)}}^2.$$

For the same edge velocities $u_\Gamma^{(i)}$ over the subdomain boundary $\partial\Omega_i$, we have $\left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}^2 \leq \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}^2$, since the infimum over a larger set is smaller. It follows that $\left|u_\Gamma^{(i)}\right|_{S_{\Gamma,E}^{(i)}}^2 \leq$

$\left|u_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}}$.

Next, we prove the connection between the edge velocity seminorms weighted by the Schur complements of the elliptic and Stokes problems for the same subdomain. Similar proof can be found in (Bramble & Pasciak, 1989).

Lemma 5.2.3. *For any $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$, we have*

$$C\frac{\beta^2}{(1+\beta)^2}\left|u_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}} \leq \left|u_\Gamma^{(i)}\right|^2_{S_{\Gamma,E}^{(i)}} \leq \left|u_\Gamma^{(i)}\right|^2_{S_\Gamma^{(i)}}$$

where β is the inf-sup stability constant defined in lemma 5.2.1.

Proof. The second inequality directly follow from the *Remark*.

We prove the first inequality as follows. Denote the discrete harmonic and Stokes extension of $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$ by $\mathcal{H}\left(u_\Gamma^{(i)}\right)$ and $\mathcal{S}\left(u_\Gamma^{(i)}\right)$, respectively. Using $v = \mathcal{S}\left(u_\Gamma^{(i)}\right) - \mathcal{H}\left(u_\Gamma^{(i)}\right)$ as the test function, by definition 5.2.7, we have

$$a\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), \mathcal{S}\left(u_\Gamma^{(i)}\right) - \mathcal{H}\left(u_\Gamma^{(i)}\right)\right) - b\left(\mathcal{S}\left(u_\Gamma^{(i)}\right) - \mathcal{H}\left(u_\Gamma^{(i)}\right), \rho\right) = 0,$$

where ρ is the corresponding pressure extension with zero mean value living in the space $W_{k-1}(\Omega_i)$.

Since $b\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), \rho\right) = 0$, it follows that $a\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), \mathcal{S}\left(u_\Gamma^{(i)}\right)\right) = a\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), \mathcal{H}\left(u_\Gamma^{(i)}\right)\right) - b\left(\mathcal{H}\left(u_\Gamma^{(i)}\right), \rho\right)$.

By Lemma 5.2.1, we have

$$\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}} \leq \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \left|\mathcal{H}^{(i)}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} + C\left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} ||\rho||_{L^2(\Omega_i)} \tag{5.56}$$

By the inf-sup condition,

$$
\begin{aligned}
\| \rho \|^2_{L^2(\Omega_i)} &\leq \beta^{-2} \sup_{v \in V_k^0(\Omega_i)} \frac{b(v,\rho)^2}{|||v|||^2} \\
&= \beta^{-2} \sup_{v \in V_k^0(\Omega_i)} \frac{a\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), v\right)^2}{|||v|||^2} \\
&\leq \beta^{-2} \left|||\mathcal{S}\left(u_\Gamma^{(i)}\right)|||^2 = \beta^{-2} \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}},
\end{aligned}
\tag{5.57}
$$

where we have used definition 5.2.7 for the second equality and Lemma 5.2.1 for the last inequality.

Substituting (5.57) into (5.56), we have

$$
\begin{aligned}
\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}} &\leq \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} + C\beta^{-1} \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \\
&\leq C\frac{1+\beta}{\beta} \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}
\end{aligned}
$$

It follows that

$$
C\frac{\beta^2}{(1+\beta)^2} \left|u_\Gamma^{(i)}\right|^2_{S_{\Gamma,S}^{(i)}} = C\frac{\beta^2}{(1+\beta)^2} \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}} \leq \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}} = \left|u_\Gamma^{(i)}\right|^2_{S_{\Gamma,E}^{(i)}}.
$$

□

In order to prove the condition number bounds for the BDDC preconditioner, we define an averaging operator for the Stokes problem, denoted by E_D, which maps $\widetilde{V}_\Gamma \times W_0$, with generally discontinuous interface velocities, to the same space with continuous interface velocities. Specifically, for any $u = (u_\Gamma, p_0) \in \widetilde{V}_\Gamma \times W_0$, $E_D\,[u_\Gamma, p_0]^T \in \widetilde{V}_\Gamma \times W_0$, where

$$
E_D = \widetilde{R}\widetilde{R}_D^T = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix} \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix} = \begin{bmatrix} E_{D,\Gamma} \\ I \end{bmatrix},
$$

and $E_{D,\Gamma} = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T$ is the interface averaging operator for the velocities across the interface

Γ. The operator $E_{D,\Gamma}$ computes a weighted average for the edge velocity across the subdomain interface Γ, and then distributes the average back to the original degree of freedoms on the interface. In our previous work, we have proved the upper bound of the operator $E_{D,\Gamma}$ for the elliptic problem

$$\begin{cases} -\Delta u = \mathrm{f} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

with hybridizable discontinuous Galerkin (HDG) discretizations(Tu & Wang, 2016). The result is cited in the following lemma.

Lemma 5.2.4. *For any $u_\Gamma \in \widetilde{V}_\Gamma$, we have*

$$|E_{D,\Gamma} u_\Gamma|^2_{\widetilde{S}_{\Gamma,E}} \leq C\gamma_{h,\tau}\left(1+\log\frac{H}{h}\right)^2 |u_\Gamma|^2_{\widetilde{S}_{\Gamma,E}},$$

where $\gamma_{h,\tau} = \max\limits_{K\in\mathcal{T}_h}\{1+\tau_K h_K\}$, τ_K and h_K are the local stabilization parameter and diameter of the mesh element K, respectively.

The proof of this lemma is based on the equivalence of norms from the bilinear form of HDG $\left(a^{HDG}(\cdot,\cdot)\right)^{1/2}$and that of the hybridized mixed method with Raviart-Thomas (RT) element $\left(a^{RT}(\cdot,\cdot)\right)^{1/2}$. Thus, the previous results of preconditioners developed for the hybrid RT methods can be applied to the HDG methods, and we proved that the upper bound for the averaging operator $E_{D,\Gamma}$ for the HDG method with high oder interface variables is similar to that of the hybridized RT method with zero oder interface variable with the addition of an extra scaling factor involving the local stabilization parameter τ_K and mesh size h_K. Importantly, a triple-bar norm of numerical trace variables is used to bridge between $\left(a^{HDG}(\cdot,\cdot)\right)^{1/2}$ and $\left(a^{RT}(\cdot,\cdot)\right)^{1/2}$. This norm is defined in(Gopalakrishnan, 2003; Cockburn et al., 2014) as below:

$$|||\hat{u}|||_D^* = \left(\frac{1}{h} \sum_{K \in \mathcal{T}_h, K \subseteq \overline{D}} ||\hat{u} - \overline{\hat{u}}||^2_{L^2(\partial K)} \right)^{1/2}$$

where $\hat{u}$ is the numerical trace of the velocity variable, $\overline{\hat{u}} = \frac{1}{|\partial K|} \int_{\partial K} \hat{u} ds$, and $|\partial K|$ is the measure of the boundary of K.

We note that the HDG and weak Galerkin discretizations are close relatives with the numerical trace of velocity $\hat{u}$ in HDG playing a similar role as the edge velocity component u_b in the weak Galerkin method. In weak Galerkin method, a fixed local stabilization parameter is chosen as $\tau_K = h_K^{-1}$, whereas this parameter τ_k may vary in HDG (Cockburn et al., 2009a; Nguyen et al., 2010). In HDG method, an auxiliary flux variable $\mathbf{q} = -\nabla u$ is introduced, which results in an expanded linear system for the elliptic problem. An additional assumption is made about the numerical trace of the flux variable on the edge/face ∂K of the mesh element as below:

$$\hat{q} \cdot n = q \cdot n + \tau_k (u - \hat{u}) .$$

With this assumption, the equivalence of the matrix forms resulting from these two methods can be established as below.

$$\begin{bmatrix} A_{\mathbf{qq}} & A_{u\mathbf{q}}^T & A_{\hat{u}\mathbf{q}}^T \\ A_{u\mathbf{q}} & A_{uu} & A_{\hat{u}u}^T \\ A_{\hat{u}\mathbf{q}} & A_{\hat{u}u} & A_{\hat{u}\hat{u}} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ u_0 \\ u_b \end{bmatrix} = \begin{bmatrix} 0 \\ (Au)_0 \\ (Au)_b \end{bmatrix},$$

where

$$\begin{aligned} &(A_{\mathbf{qq}}\mathbf{q}, \mathbf{v}) = (\mathbf{q}, \mathbf{v})_{\mathcal{T}_h}, \qquad (A_{u\mathbf{q}}\mathbf{q}, w) = (\nabla \cdot \mathbf{q}, w)_{\mathcal{T}_h}, \quad \left(A_{\hat{u}\mathbf{q}}\mathbf{q}, \mu\right) = -\langle \mathbf{q} \cdot \mathbf{n}, \mu \rangle_{\partial \mathcal{T}_h}, \\ &(A_{uu}u, w) = \langle \tau_k u, w \rangle_{\partial \mathcal{T}_h}, \quad (A_{\hat{u}u}u, \mu) = -\langle \tau_k u, \mu \rangle_{\partial \mathcal{T}_h}, \quad (A_{\hat{u}\hat{u}}\hat{u}, \mu) = \langle \tau_k \hat{u}, \mu \rangle_{\partial \mathcal{T}_h}, \end{aligned}$$

for all $(\mathbf{q}, \{u, \hat{u}\}) \in \left(\mathbf{Q}_{k-1}, V_k^0\right)$ and $(\mathbf{v}, \{w, \mu\}) \in \left(\mathbf{Q}_{k-1}, V_k^0\right)$. The subscripts 0 and b in-

dicate the restriction of the vector Au to the components corresponding to the interior and edge parts, respectively. A is the operator for the elliptic bilinear form of the weak Galerkin method, as defined in (5.47), and $u = \{u_0, u_b\}$ is the weak velocity function. When the polynomial approximation spaces with the same degree order are used for the variable counerparts of these two methods, and τ_K is set to be h_K^{-1}, it is clear that A matrix resulting from the weak Galerkin method is the Schur complement formed by eliminating the variables of the leading block $A_{\mathbf{qq}}$ of the HDG matrix. This reveals the connection between these two methods.

Weak Galerkin methods boasts flexibility of polynomial space selection in practical computation (Mu et al., 2015b). In this work, we use a combination of polynomial spaces designed to balancing the conflicting requirements between computational cost and accuracy(Mu et al., 2015b). Take the scalar elliptic problem for example, in weak Galerkin method, we use $((\nabla_w), u_0, u_b) \in \left([P_{k-1}]^d, P_k, P_{k-1}\right)$.Rigorous analysis on the connection between the triple-bar norm $|||\cdot|||^*$ and the norm induced by the bilinear form of the weak Galerkin method $|||\cdot|||$ was provided in Lemma 4.2.2. We cite Lemma 4.2.3, which proves the bound of the averaging operator for weak Galerkin method, as below.

Lemma 5.2.5. *For any $u_\Gamma \in \widetilde{V}_\Gamma$, we have*

$$|E_{D,\Gamma} u_\Gamma|^2_{\widetilde{S}_{\Gamma,E}} \leq C \left(1 + \log \frac{H}{h}\right)^2 |u_\Gamma|^2_{\widetilde{S}_{\Gamma,E}},$$

for the weak Galerkin discretization.

Now, we are in a position to prove the bound of the averaging operator E_D for the Stokes problem.

Lemma 5.2.6. *There exists a positive constant C, which is independent of H and h, such that*

$$|E_D w|^2_{\widetilde{S}} \leq C \left(\frac{1+\beta}{\beta}\right)^2 \left(1 + \log \frac{H}{h}\right)^2 |w|^2_{\widetilde{S}} \qquad \forall w = (w_\Gamma, q_0) \in \widetilde{V}_{\Gamma,B} \times W_0,$$

where β is the inf-sup stability constant.

Proof. For any vector $w = (w_\Gamma, q_0) \in \widetilde{V}_{\Gamma,B} \times W_0$, by Lemma 5.2.2, $\widetilde{R}_D^T w \in \widehat{V}_{\Gamma,B} \times W_0$. Thus, $E_D w = \widetilde{R}\widetilde{R}_D^T w \in \widetilde{V}_{\Gamma,B} \times W_0$.

From the definition of the $\widetilde{S}$-seminorm, we have

$$|E_D w|_{\widetilde{S}}^2 = \left\| E_{D,\Gamma} w_\Gamma \right\|_{\widetilde{S}_\Gamma}^2 = |\bar{R}_\Gamma \left(E_{D,\Gamma} w_\Gamma \right)|_{S_\Gamma}^2.$$

Noting that $S_\Gamma = diag(S_\Gamma^{(i)})$, and applying Lemma 5.2.3 to each subdomain, we have
$|\bar{R}_\Gamma \left(E_{D,\Gamma} w_\Gamma \right)|_{S_\Gamma}^2 \leq C \left(\frac{1+\beta}{\beta} \right)^2 |\bar{R}_\Gamma \left(E_{D,\Gamma} w_\Gamma \right)|_{S_{\Gamma,E}}^2$.
Further, we have
$|\bar{R}_\Gamma \left(E_{D,\Gamma} w_\Gamma \right)|_{S_{\Gamma,E}}^2 = |E_{D,\Gamma} w_\Gamma|_{\widetilde{S}_{\Gamma,E}}^2 \leq C \left(1 + \log \frac{H}{h} \right)^2 |w_\Gamma|_{\widetilde{S}_{\Gamma,E}}^2 \leq C \left(1 + \log \frac{H}{h} \right)^2 |w_\Gamma|_{\widetilde{S}_\Gamma}^2$.
Combining these inequalities, we have
$|E_D w|_{\widetilde{S}}^2 \leq C \left(\frac{1+\beta}{\beta} \right)^2 \left(1 + \log \frac{H}{h} \right) |w_\Gamma|_{\widetilde{S}_\Gamma}^2 = C \left(\frac{1+\beta}{\beta} \right)^2 \left(1 + \log \frac{H}{h} \right)^2 |w|_{\widetilde{S}}^2$.

□

5.2.6 Condition number estimate for the BDDC preconditioner

We are now ready to formulate and prove our main results. It follows by proving the lower and upper bound for $u^T M^{-1} \hat{S} u$. See similar proof in (Li & Widlund, 2006a; Mandel et al., 2005; Tu, 2006, 2007d,c).

Theorem 5.2.7. *Assume the divergence free constraint holds for the interface velocities. The preconditioned operator $M^{-1}\hat{S}$ is symmetric, positive definite with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\widehat{S}}$ on the space $\widehat{V}_{\Gamma,B} \times W_0$. Its eigenvalues are bounded from below by 1 and from above by $C \frac{(1+\beta)^2}{\beta^2} \left(1 + \log \frac{H}{h} \right)^2$, where C is a constant which is independent of H, h, and the number of subdomains, and β is the inf-sup stability constant.*

Proof. It is sufficient to prove that for any $u = (u_\Gamma, p_0) \in \hat{V}_{\Gamma,B} \times W_0$, with $u_\Gamma \neq 0$,
$\langle u, u \rangle_{\hat{S}} \leq \left\langle u, M^{-1} \hat{S} u \right\rangle_{\hat{S}} \leq C \left(\frac{1+\beta}{\beta} \right)^2 \left(1 + \log \left(\frac{H}{h} \right) \right)^2 \langle u, u \rangle_{\hat{S}}$.

In what follows, we prove the lower and upper bound for $\left\langle u, M^{-1}\hat{S}u\right\rangle_{\hat{S}}$ respectively.

Let $\tilde{u} = \tilde{S}^{-1}\tilde{R}_D\hat{S}u$. Obviously, $\tilde{u} \in \tilde{V}_{\Gamma,B} \times W_0$.

Note that $\widetilde{R}^T\widetilde{R}_D = \widetilde{R}_D^T\widetilde{R} = I$. The details for the proof of the lower bound go as follows:

$$\begin{aligned}\langle u, u\rangle_{\hat{S}} = u^T\hat{S}\widetilde{R}_D^T\widetilde{R}u = u^T\hat{S}\widetilde{R}_D^T\tilde{S}^{-1}\tilde{S}\widetilde{R}u = \left\langle \tilde{u}, \widetilde{R}u\right\rangle_{\widetilde{S}} \\ \leq \langle\tilde{u},\tilde{u}\rangle_{\widetilde{S}}^{1/2}\left\langle\widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{S}}^{1/2} = \langle\tilde{u},\tilde{u}\rangle_{\widetilde{S}}^{1/2}\langle u,u\rangle_{\hat{S}}^{1/2}.\end{aligned}$$

Thus, we obtain $\langle u, u\rangle_{\hat{S}} \leq \langle\tilde{u},\tilde{u}\rangle_{\tilde{S}}$ by cancelling a common factor and squaring on both sides.

Since

$\langle\tilde{u},\tilde{u}\rangle_{\tilde{S}} = u^T\hat{S}\widetilde{R}_D^T\tilde{S}^{-1}\tilde{S}\tilde{S}^{-1}\tilde{R}_D\hat{S}u = \left\langle u, \widetilde{R}_D^T\tilde{S}^{-1}\tilde{R}_D\hat{S}u\right\rangle_{\hat{S}} = \left\langle u, M^{-1}\hat{S}u\right\rangle_{\hat{S}}$,

we have $\langle u, u\rangle_{\hat{S}} \leq \left\langle u, M^{-1}\hat{S}u\right\rangle_{\hat{S}}$.

Next, we prove the upper bound.

Since $M^{-1} = \widetilde{R}_D^T\tilde{S}^{-1}\widetilde{R}_D$, we have $\tilde{R}_D^T\tilde{u} = M^{-1}\hat{S}u$.

By using Lemma 5.2.6 and the fact that $\hat{S} = \tilde{R}^T\tilde{S}\tilde{R}$, we obtain

$$\begin{aligned}&\left\langle M^{-1}\hat{S}u, M^{-1}\hat{S}u\right\rangle_{\hat{S}} = \left\langle\tilde{R}_D^T\tilde{u}, \tilde{R}_D^T\tilde{u}\right\rangle_{\hat{S}} = \left\langle\tilde{R}\tilde{R}_D^T\tilde{u}, \tilde{R}\tilde{R}_D^T\tilde{u}\right\rangle_{\tilde{S}} \\ &= |E_D\tilde{u}|_{\tilde{S}}^2 \leq C\left(\frac{1+\beta}{\beta}\right)^2\left(1+\log\frac{H}{h}\right)^2|\tilde{u}|_{\tilde{S}}^2 \leq C\left(\frac{1+\beta}{\beta}\right)^2\left(1+\log\frac{H}{h}\right)^2\left\langle u, M^{-1}\hat{S}u\right\rangle_{\hat{S}}\end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left\langle u, M^{-1}\hat{S}u\right\rangle_{\hat{S}} &\leq \langle u,u\rangle_{\hat{S}}^{1/2}\left\langle M^{-1}\hat{S}u, M^{-1}\hat{S}u\right\rangle_{\hat{S}}^{1/2} \\ &\leq C\frac{1+\beta}{\beta}\left(1+\log\frac{H}{h}\right)\langle u,u\rangle_{\hat{S}}^{1/2}\left\langle u, M^{-1}\hat{S}u\right\rangle_{\hat{S}}^{1/2}.\end{aligned}$$

This gives $\left\langle u, M^{-1}\hat{S}u\right\rangle_{\hat{S}} \leq C\left(\frac{1+\beta}{\beta}\right)^2\left(1+\log\frac{H}{h}\right)^2\langle u,u\rangle_{\hat{S}}$. The upper bound of the eigenvalues thus follows.

Number of Subdomains	Iterations	Condition number
4×4	11	4.12
8×8	13	5.01
12×12	13	4.77
16×16	13	4.90
20×20	13	5.05

Table 5.4: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, and $k = 1$.

Number of Subdomains	Iterations	Condition number
4×4	13	7.37
8×8	17	9.24
12×12	20	9.41
16×16	20	9.89
20×20	20	10.17

Table 5.5: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$, and $k = 2$.

□

5.2.7 Numerical Experiments

In this subsection, we will report some numerical results for the BDDC algorithm proposed for the weak Galerkin discretization of the Stokes problem. We used the BDDC algorithm to solve the model problem (5.44) on the square domain $\Omega = [0, 1]^2$ with zero Dirichlet boundary condition. The analytical solution of the test problem is given by

$$u = \begin{bmatrix} sin^3(\pi x)\, sin^2(\pi y)\, cos(\pi y) \\ -sin^2(\pi x)\, sin^3(\pi y)\, cos(\pi x) \end{bmatrix} \quad \text{and} \qquad p = x^2 - y^2.$$

We decompose the unit square into $N \times N$ subdomains with side length $H = 1/N$. Each subdomain has a characteristic mesh size h. Both the first order ($k = 1$) and second order ($k = 2$) weak Galerkin methods are used to discretize the model equations. The BDDC preconditioned conjugate gradient iterations are stopped when the l_2-norm of the residual has been reduced by a factor of 10^6.

$\frac{H}{h}$	Iterations	Condition number
4	9	2.49
8	13	5.01
12	15	6.42
16	15	7.48
20	16	8.36

Table 5.6: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, and $k = 1$.

$\frac{H}{h}$	Iterations	Condition number
4	14	5.87
8	17	9.24
12	19	11.57
16	21	12.47
20	21	13.58

Table 5.7: Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, and $k = 2$.

In the first set of experiments, we fix the size of the subdomain problem to be $\frac{H}{h} = 8$. Table 5.4 and 5.5 show the iteration counts and the estimates of the condition numbers for the BDDC preconditioned operator with changing subdomain numbers for $k = 1$ and $k = 2$, respectively. The condition numbers are found to be independent of the number of subdomains. As another set of experiment, instead of fixing the size of the subdomain problems, we fix the subdomain partition to be 8×8, and allow the subdomain problem size to vary. The condition number is found to increase logarithmically with the subdomain problem size. Table 5.6 and 5.7 demonstrate results for the second set of experiments for $k = 1$ and $k = 2$, respectively.

To conclude, we have carried out a series of experiments to obtain iteration counts and condition number estimates. The experimental results prove to be consistent with the theory. That is the condition number bound of the BDDC preconditioned system is of the form $C\frac{(1+\beta)^2}{\beta^2}\left(1+\log\frac{H}{h}\right)^2$, where H and h are the diameters of the subdomains and elements, respectively. Possible future work will be to explore the high order effects on C.

Chapter 6

Concluding Remarks

We have designed BDDC preconditioners for linear systems arising from HDG and WG discretizations of Poisson and Stokes equations. The condition number bound of the preconditioned operator is shown to be polylogarithmically dependent on the size of the subdomain problem, which is consistent with results for elliptic problem using standard finite element discretizations. We have also conducted numerical experiments to validate the theoretical analysis. The numerical observations agree well with the theoretical results. Possible future work will be to study the dependence of condition number bound with the order of approximation polynomials, and to develop overlapping domain decomposition preconditioners for linear systems of our interest.

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Appendix A

Shape Regularity Assumptions of the Mesh

Let $\mathcal{T}_h$ be a shape-regular tessellation of Ω with polygons in 2D and polyhedra in 3D. We denote the element in $\mathcal{T}_h$ by T, the diameter of T by h_T, and the area/volume of T by $|T|$. The mesh size is characterized by $h \coloneqq max_{T\in\mathcal{T}_h} h_T$. Define $\mathcal{F}_h$ be the set of edges/faces of elements $T \in \mathcal{T}_h$. $\mathcal{F}_h^i$ and $\mathcal{F}_h^\partial$ are subsets of $\mathcal{F}_h$, which consists of domain interior and boundary edges, respectively. We denote by $|e|$ the length/area of e and h_e the diameter of the edge/face in $\mathcal{F}_h$.

The following shape regularity assumptions are needed for the finite element partition in order to have the desired approximation properties in the weak Galerkin finite element space.

- There exist positive constants ϱ_T and ϱ_e such that

$$\varrho_T\, h_T^d \le |T|, \quad \varrho_e\, h_e^{d-1} \le |e|, \qquad \forall T \in \mathcal{T}_h,\ e \in \mathcal{F}_h.$$

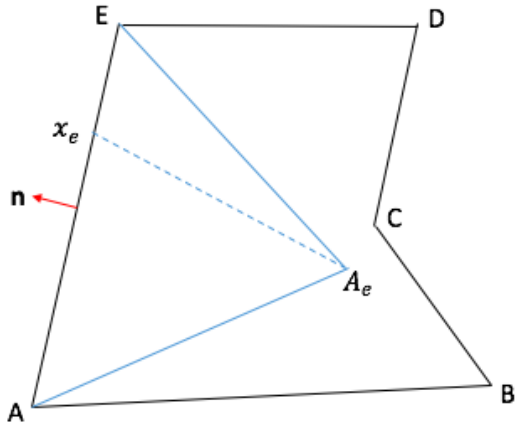

Figure A.1: An example of a shape-regular polygonal element ABCDE

- There exists a positive constant κ such that

$$\kappa h_T \leq h_e, \qquad \forall T \in \mathcal{T}_h,\ e \in \mathcal{F}_h.$$

- Assume the mesh elements have linear edges/faces. For each $T \in \mathcal{T}_h$, $e \in \mathcal{F}_h$, there exists a pyramid $P(e, T, A_e)$ contained in T with its base e and apex A_e. The height of the pyramid is given by $\sigma_e h_T$ with $\sigma_e \geq \sigma_* > 0$ for some fixed positive number σ_*. Besides, the angle between the vector $x_e - A_e$ for any $x_e \in e$, and the outward normal direction of e is strictly acute.

- Assume that each $T \in \mathcal{T}_h$ has a circumscribing simplex $S(T)$ that is shape regular and has a diameter $h_{S(T)} \leq \gamma_* h_T$ with γ_* being a positive constant independent of T. Further, each circumscribed simplex $S(T)$ intersects with a small and fixed number of such simplices for all other $T \in \mathcal{T}_h$.

Figure A.1 illustrates a shape regular polygonal element. Under the above shape regularity assumptions of the mesh, we have the trace and inverse inequalities (1.5.3) and (1.5.4); see(Wang & Ye, 2014) for details.

Appendix B

Lagrange Element

We used the Lagrange triangle in the simulation. The nodal basis functions are tabulated below.

Table B.1: Nodal basis $\{\phi_1, \phi_2, ..., \phi_k\}$ of shape function space

Linear Lagrange triangle	Quadratic Lagrange triangle
$\phi_1 = 1 - \xi - \eta$	$\phi_1 = (1 - \xi - \eta)(1 - 2\xi - 2\eta)$
$\phi_2 = \xi$	$\phi_2 = \xi(2\xi - 1)$
$\phi_3 = \eta$	$\phi_3 = \eta(2\eta - 1)$
	$\phi_4 = 4\xi\eta$
	$\phi_5 = 4\eta(1 - \xi - \eta)$
	$\phi_6 = 4\xi(1 - \xi - \eta)$

More details can be found in (Brenner & Scott, 2008).